Lecture (11) 19/2/19.

# $\bar{\partial}$ derivative

$\bar{\partial}$ is pronounced as DBAR.

Let us pause and make a very important comment here. While it may be convenient to work w/ "analytic" f$^n$'s; this by no way means that "non-analytic" f$^n$'s are merely mathematical artifacts. Infact there are several generalizations of theorems that extend to the "non-analytic" case (like the one stated below) which are used in the study of nonlinear wave propagation through the use of scattering and inverse scattering theories.

$z = x + iy$ and $\bar{z} = x - iy$ Can be rewritten as

$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

Apply chain rule & total derivative $\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}$

$$\Rightarrow \boxed{\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)} \quad -(1)$$

you may think of $z$ and $\bar{z}$ as being treated as independent variables here in the sense that $\frac{dz}{d\bar{z}} = 0$

and hence

Def$^n$. of DBAR derivative

$$\boxed{\bar{\partial} \equiv \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)} \quad -(2)$$

Eqs (1) & (2) are just definitions of 2 differential operators.

Notation :- In general, we may choose to write $f \equiv f(z, \bar{z})$.

But if $f$ is differentiable in $z$ & $\bar{z}$ and $\frac{\partial f}{\partial \bar{z}} = 0$ ; then simply say $f = f(z)$.

Hey! But shouldn't CR eqns imply $\frac{\partial f}{\partial \bar{z}} \equiv 0$ ?

Yes, but CR eqns apply only in the case when $f$ is analytic, whence $\frac{\partial f}{\partial \bar{z}} \equiv 0$

but the same is not true for any general non-analytic $f^n$ !!

Very important

Let us verify this!

Let $f = u + iv$

Using (2): $$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv)$$

$$= \frac{1}{2}\left\{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right\}$$

If $f$ analytic, then $u_x = v_y$, $u_y = -v_x$ $\Rightarrow$ $= 0 + i0 = 0$

So this is another way to check analyticity of a $f^n$. i.e. chk $\frac{\partial f}{\partial \bar{z}} = 0$.

Green's th$^m$

① $$\oint_C u\,dx + v\,dy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy \quad \text{—(3)}$$

Where $R$ is a simply connected region bdd by a Jordan contour $C$

here- $u = g$, $v = ig$

(II) $$\oint_C g\,d\zeta = 2i \iint_R \frac{\partial g}{\partial \bar{\zeta}}\, dA(\zeta) \quad \text{—(4)}$$

where $\zeta = \xi + i\eta$, $d\zeta = d\xi + i\,d\eta$ and $dA(\zeta) = d\xi\, d\eta$

and $$\frac{\partial g}{\partial \bar{\zeta}} = \frac{1}{2}\left(\frac{\partial g}{\partial \xi} + i\frac{\partial g}{\partial \eta}\right)$$

**Thm(11.1): (Generalized Cauchy Integral Formula).**

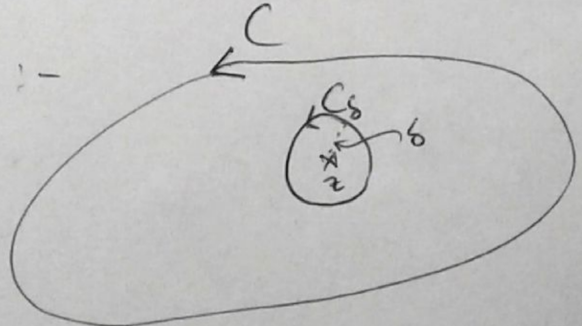
If $\frac{\partial f}{\partial \bar{\zeta}}$ exists and is continuous in a region $R$ bounded by a Jordan contour $C$; then at any interior point $z$

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)}\, d\zeta - \cancel{\frac{1}{\pi}\iint_R \frac{\frac{\partial f}{\partial \bar{\zeta}}}{(\zeta - z)}\, dA(\zeta)} \quad \text{—(5)}$$

0 only when $f$ is analytic in which case eq(5) is known as the Cauchy's Integral Formula.

We will prove this version

Proof:-




Inside the contour $C$, inscribe a small circle $C_\delta$ w/ rad $= \delta$ and center at $z$

(Recall: We have studied this kind of deformation of contour in the prev. lecture).

From examples of application of Cauchy's th$^m$ (Cauchy-Goursat) We can deform the contour $C$ into $C_\delta$ s.t.

$$\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta = \oint_{C_\delta} \frac{f(\zeta)}{(\zeta - z)}\, d\zeta$$

$$= f(z)\underbrace{\oint_{C_\delta}\frac{d\zeta}{(\zeta - z)}} + \oint_{C_\delta}\frac{f(\zeta)-f(z)}{(\zeta - z)}\,d\zeta$$

use $(\zeta - z) = \delta e^{i\theta}$ to get

$$\int_0^{2\pi}\frac{i\delta e^{i\theta}}{\delta e^{i\theta}}\,d\theta = 2\pi i$$

B/c $f(z)$ is continuous (why? analyticity $\Rightarrow$ continuity)

$$|f(\zeta) - f(z)| < \epsilon \quad \text{for} \quad |\zeta - z| < \delta \ (\text{small})$$

$$\therefore \left|\oint_{C_\delta}\frac{f(\zeta)-f(z)}{\zeta - z}\,d\zeta\right| \le \oint_{C_\delta}\frac{|f(\zeta)-f(z)|}{|\zeta - z|}\,|d\zeta|$$

$$< \oint_{C_\delta}\frac{\epsilon}{\delta}\,|d\zeta| = 2\pi\epsilon \xrightarrow[\text{as } \epsilon \to 0]{} 0$$

b/c $\oint_{C_\delta}|d\zeta| = 2\pi\delta$ (circumference of $C_\delta$).

$$\therefore \oint_C \frac{f(\zeta)}{(\zeta - z)}\,d\zeta = f(z)\,2\pi i$$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\oint\frac{f(\zeta)}{(\zeta - z)}\,d\zeta$$

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$Th^m$ (11.2) : If $f(z)$ is analytic interior to and on a Jordan contour C, then all derivatives $f^{(K)}(z)$, $K = 1, 2, 3, \ldots$ exist in the domain D interior to C and

$$f^{(K)}(z) = \frac{K!}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta - z)^{K+1}}\, d\zeta .$$

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We will not prove this $Th^m$ (you may chk pg 93 of Textbook).

NOTE :- (I) The consequences of $Th^m$(11.1) and $Th^m$(11.2) are remarkable b/c they show that if a (an analytic) $f^n$ $f(z)$ is known on the Jordan contour C ; then values of the $f^n$ & all its derivatives can be found at all pts inside C.

This is again a very strong principle that shows up in many areas of mathematics & physics whereby we find that almost all of the information abt. the "universe" (a domain) is found to be localized entirely on the boundary of the universe/domain. I had remarked this earlier as well during our discussion on the proof of the Cauchy-Goursat $Th^m$.

(II) Moreover, if a $f^n$ is analytic (has 1st derivative) in D then all its higher derivatives exist in D. This is not necessarily true for $f^n$s of real variables. (eg. try $f(x) = x^{3/2}$).

(iii) Additionally, if $|\zeta - z| = R$ and $|f(z)| < M$

then $$|f^{(n)}(z)| \le \frac{n!}{2\pi} \oint_C \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} |d\zeta|$$

$$\le \frac{n!\,M}{2\pi R^{n+1}} \oint_C |d\zeta|$$

$$\le \frac{n!\,M}{R^n} \quad \text{b/c } \oint_C |d\zeta| \le 2\pi R.$$

<u>$Th^m$ (11.2)</u> (<u>Liouville</u>) If $f(z)$ is entire & bdd in the extended $z$-plane; then $f(z) = \text{Const}$.

<u>Proof</u> :- $$|f'(z)| \le \frac{M}{R}$$

B/c the above is true for any pt. $z$ in the plane $\Rightarrow$ R can be made arbitrarily large $\Rightarrow f'(z) = 0$ for any pt $z$ in the plane.

Fundamental $Th^m$ of calculus $\Rightarrow f(z) - f(0) = \int_0^z f'(\zeta)\,d\zeta = 0$

$\Rightarrow f(z) = f(0) = \text{const}$

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Converse of Cauchy's th$^m$ ($f$ analytic $\Rightarrow \oint_C f dz = 0$)
is known as Morera's th$^m$

th$^m$ (11·4) (Morera)

If $f$ is continuous in a domain $D$ & if $\oint_C f(z)\,dz = 0$ for every Jordan Contour $C$ in $D$; then $f(z)$ is analytic in $D$.

Proof:- $f$ continuous & $\oint_C f\,dz = 0 \Rightarrow \exists$ a $f^n$ $F(z)$ that is analytic in $D$ s.t. $F'(z) = f(z)$

Do Not prove this

(Why? See book, pg 84)

Now th$^m$ (11.2) $\Rightarrow F'(z) = f(z)$ is analytic if $F(z)$ is analytic.

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th$^m$ (11.5) (Gauss Mean value th$^m$)

Let $f(z)$ be analytic in $|z-a| \leq R$. then $\langle f(z) \rangle_{C_R} = f(a)$ where $\langle\,\rangle_{C_R}$ means avg. value computed on $|z-a| = R$.

for proof; see the relevant question in HW(3)