Lecture (13): Proof of Laurent Series theorem.

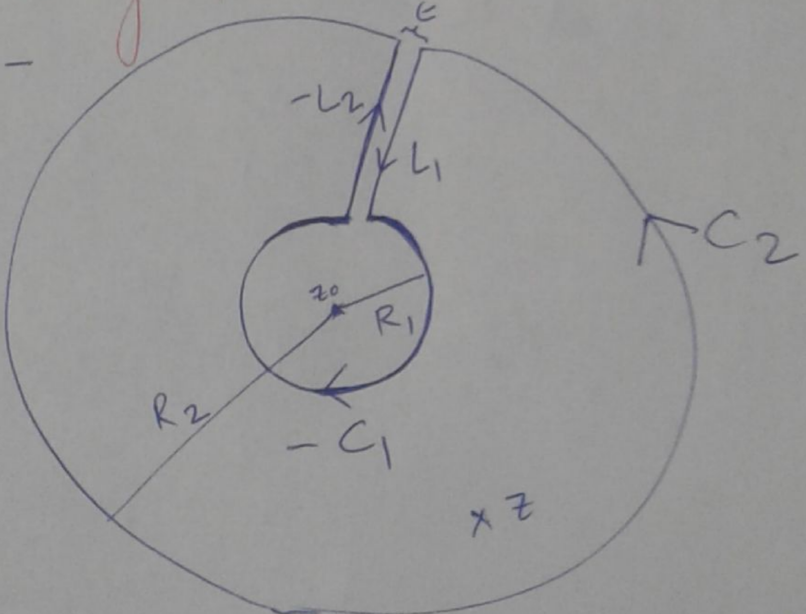
Thm (13.1) (Laurent Series):– A f^n $f(z)$ analytic in an annulus $R_1 \le |z - z_0| \le R_2$ may be represented by the expansion

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n \qquad \text{——— (13.1)}$$

in the region $R_1 < R_a \le |z - z_0| \le R_b < R_2$,

where $C_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$ —(13.2) and C is any simple closed (Jordan) contour in the region of analyticity enclosing the inner boundary $|z - z_0| = R_1$

Proof:–



Introduce the usual crosscut in the annulus s.t. C_1 and C_2 lie on $|z - z_0| = R_1$ and $|z - z_0| = R_2$ respectively as shown in the figure above.

Define $\tilde{C} = C_2 + L_1 - C_1 - L_2$ as the Jordan contour.

* ζ = pronounced "zeta"
ξ = pronounced "xi"
χ = pronounced "chi"

Apply the Cauchy integral formula to any interior point enclosed by $\tilde{C}$

$$f(z) = \frac{1}{2\pi i}\oint_{\tilde{C}} \frac{f(\zeta)}{(\zeta - z)}\, d\zeta$$

b/c $f(z)$ is analytic in the region bdd by $\tilde{C}$.

$$= \frac{1}{2\pi i}\left\{ \int_{C_2} \frac{f(\zeta)}{(\zeta - z)}\, d\zeta + \int_{L_1} \frac{f(\zeta)}{(\zeta - z)}\, d\zeta - \int_{L_2} \frac{f(\zeta)\, d\zeta}{(\zeta - z)} - \int_{C_1} \frac{f(\zeta)}{(\zeta - z)}\, d\zeta \right\}$$

Now slowly stretch the contour C in such a way that $\epsilon \to 0$ while keeping the radius of "circle" formed by C_2 and C_1 fixed at R_2 and R_1 respectively.

$$f(z) = \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta - z)}\, d\zeta - \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - z)}\, d\zeta \quad \text{—(13.3)}$$

$$= \frac{1}{2\pi i}\left\{ \oint_{C_2} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0) - (z - z_0)} + \oint_{C_1} \frac{f(\zeta)\, d\zeta}{(z - z_0) - (\zeta - z_0)} \right\}$$

$$= \frac{1}{2\pi i}\left\{ \oint_{C_2} \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)\left(1 - \frac{(z - z_0)}{(\zeta - z_0)}\right)} + \oint_{C_1} \frac{f(\zeta)\, d\zeta}{(z - z_0)\left(1 - \frac{(\zeta - z_0)}{(z - z_0)}\right)} \right\}$$

We have used $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$ for $|z|<1$

$$= \frac{1}{2\pi i}\left\{ \oint_{C_2} \frac{f(\zeta)}{(\zeta - z_0)} \sum_{j=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^j d\zeta + \oint_{C_1} \frac{f(\zeta)}{(z - z_0)} \sum_{j=0}^{\infty} \left(\frac{\zeta - z_0}{z - z_0}\right)^j d\zeta \right\}$$

We can swap the integral and the summation by using Th^m (12.1) and noting that "sequence" and "series" are equivalent ideas.

$$f(z) = \sum_{j=0}^{\infty} A_j (z-z_0)^j + \sum_{j=0}^{\infty} B_j (z-z_0)^{-(j+1)} \quad \text{——} (13.4)$$

where

$$\left.\begin{aligned} A_j &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta-z_0)^{j+1}}\,d\zeta \\ B_j &= \frac{1}{2\pi i}\oint_{C_1} f(\zeta)(\zeta-z_0)^j\,d\zeta \end{aligned}\right\} \quad \text{——} (13.5)$$

Next, we let $n = j$ in the first sum
& $n = -(j+1)$ in the 2nd sum above.

$$f(z) = \sum_{n=0}^{\infty} A_n (z-z_0)^n + \sum_{n=-\infty}^{-1} B_{-n-1} (z-z_0)^n \quad \text{——} (13.6)$$

$$\text{Where} \left.\begin{aligned} A_n &= \frac{1}{2\pi i}\oint_{C_2} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta \\ \& \quad B_{-n-1} &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta \end{aligned}\right\} \quad \text{——} (13.7)$$

B/c $f(z)$ is analytic in the annulus, we can deform in each of C_2 and C_1 ~~both~~ the integrals in (13.7) to obtain $\frac{1}{2\pi i}\oint_C \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}$ and call it

$$C_n = A_n \quad \forall\, n \geq 0$$
$$C_n = B_{-n-1} \quad \forall\, n \leq -1$$

Here C is any Jordan contour lying entirely in the annulus & enclosing C_1

this enables us to write (13.6) succinctly as $f(z) = \sum_{n=-\infty}^{\infty} C_n (z-z_0)^n$ w/

$$C_n = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}} \quad \#$$

the "uniform convergence" of the Laurent series given by eq (13.1) to $f(z)$ can be verified as follows.

Consider the version of the L.S. given by eq (13.6)

$$f(z) = \underbrace{\sum_{n=0}^{\infty} A_n (z-z_0)^n}_{f_1(z)} + \underbrace{\sum_{n=-\infty}^{-1} B_{-n-1} (z-z_0)^n}_{f_2(z)}$$

this is simply the Taylor series part of the L.S. w/ +ve powers n & We know that the Taylor series converges uniformly to $f_1(z)$.

Here we will apply the Weierstrass M-test.

Recall from (13.4) that $f_2(z) = \sum_{j=0}^{\infty} B_j (z-z_0)^{-(j+1)}$.

For j large enough & $z = z_1$ on $|z-z_0| = R_1$

$$\left|B_j (z-z_0)^{-(j+1)}\right| = \frac{|B_j|}{|z_1-z_0|^{j+1}} \left|\frac{z_1-z_0}{z-z_0}\right|^{j+1} < \underbrace{\left|\frac{z_1-z_0}{z-z_0}\right|^{j+1}}_{M_j} < 1$$

< 1 b/c of the conv. of the series for $f_2(z)$ above.

< 1 b/c z_1 is on C_1 & z is as shown in fig in pg(1)

$\therefore \sum_{j=0}^{\infty} \left(\frac{z_1-z_0}{z-z_0}\right)^{j+1} < \infty$

$\Rightarrow f_2(z)$ conv. unif. by M-test. #