7. Definition (<u>Linear transformations</u>):

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a *linear transformation* if $\exists\ A \in \mathbb{M}_{m\times n}(\mathbb{R})$ such that $T(\mathbf{x}) = \mathbf{Ax},\ \forall \mathbf{x} \in \mathbb{R}^{\mathbf{n}}$.

eg. The rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is a linear transformation which rotates a vector in $\mathbb{R}^2$ by θ.

Ques: Given $T : \mathbb{R}^n \to \mathbb{R}^m$, how do we find A?

Ans: $A = \begin{pmatrix} | & | & & & | \\ T(\mathbf{e_1}) & T(\mathbf{e_2}) & \cdot & \cdot & T(\mathbf{e_n}) \\ | & | & & & | \end{pmatrix}$ *where $\mathbf{e_i}$ is the i^{th} standard basis element of $\mathbb{R}^n$.*

A square matrix is <u>*invertible*</u> *if its linear transformation is invertible.*

Theorem: A $n \times n$ matrix A is invertible $\iff rref(A) = I_n \equiv rank(A) = n$.

Finding inverse of a matrix: $A \in \mathbb{M}_{n\times n}(\mathbb{R})$. In order to find A^{-1}, form the augmented matrix $\tilde{A} = \begin{pmatrix} A & | & I_n \end{pmatrix}$ and compute $rref(\tilde{A})$.

- If $rref(\tilde{A})$ is of the form $\begin{pmatrix} I_n & | & B \end{pmatrix}$, then $A^{-1} = B$.
- If $rref(\tilde{A})$ is of another form, then A is <u>not</u> invertible.

$(AB)^{-1} = B^{-1}A^{-1}$.