8. Definition (Image or range of a matrix/linear transformation):

$Im(A) = Im(T)$ is the *span* of the column vectors of A.

Q) Find a basis of the image of $A = \begin{pmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{pmatrix} = \begin{pmatrix} | & | & | & | & | \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ | & | & | & | & | \end{pmatrix}$ and determine $dim(Im(A))$.

Ans) To find the basis of Im(A), we need to identify the redundant columns of A from amongst all the column vectors of A. By inspection of A, it will be hard to tell which of the columns of A are redundant (linearly dependent on the others). So we will transform A to B = rref(A).

$$B = rref(A) = \begin{pmatrix} 1 & 2 & 0 & 3 & -4 \\ 0 & 0 & 1 & -4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} | & | & | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \\ | & | & | & | & | \end{pmatrix}.$$

The redundant columns of B correspond to the redundant columns of A. The redundant columns of B are also easy to spot: *They are the columns that do not contain a leading 1*, namely, $\mathbf{b}_2 = 2\mathbf{b}_1$, $\mathbf{b}_4 = 3\mathbf{b}_1 - 4\mathbf{b}_3$, and $\mathbf{b}_5 = -4\mathbf{b}_1 + 5\mathbf{b}_3$. Thus the redundant columns of A are $\mathbf{a}_2 = 2\mathbf{a}_1$, $\mathbf{a}_4 = 3\mathbf{a}_1 - 4\mathbf{a}_3$, and $\mathbf{a}_5 = -4\mathbf{a}_1 + 5\mathbf{a}_3$. And the non-redundant columns of A are $\mathbf{a}_1$ and $\mathbf{a}_3$, they form a basis of image of A. Therefore, a basis of image of A is

$$\begin{pmatrix} 1 \\ -1 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 5 \\ 1 \end{pmatrix}$$

$dim(Im(A)) = 2.$

9. **Definition ([Kernel of T (or equivalently the null space of A, $Null(A)$)]):** The set of all $x \in \mathbb{R}^n$ s.t. $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$.

Q) Find a basis of the kernel of A (equivalently, $Null(A)$) and determine $dim(Ker(A)) = dim(null(A))$.

Ans) Most importantly $Ker(A) = Ker(rref(A)) = Ker(B)$. So we might as well solve for $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ s.t. $B\mathbf{x} = \mathbf{0}$. This is done by considering the augmented matrix $\tilde{B} = \begin{pmatrix} B & | & \mathbf{0} \end{pmatrix}$ from which we have the following:

$$x_1 + 2x_2 + 0x_3 + 3x_4 - 4x_5 = 0$$
$$0x_1 + 0x_2 + x_3 - 4x_4 + 5x_5 = 0$$

or equivalently,

$$x_1 = -2x_2 - 3x_4 + 4x_5$$
$$x_3 = 4x_4 - 5x_5$$

whence $x_2 = \alpha, \quad x_4 = \beta, \quad x_5 = \gamma$ are set arbitrarily. Therefore,

$$\mathbf{x} = \begin{pmatrix} -2\alpha - 3\beta + 4\gamma \\ \alpha \\ 4\beta - 5\gamma \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} -2\alpha & -3\beta & +4\gamma \\ \alpha & & \\ & 4\beta & -5\gamma \\ & \beta & \\ & & \gamma \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix}.$$

The $Null(A)$ is spanned by these basis vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{pmatrix}$ and $dim(Null(A)) = 3.$

Exercise problem: Find the basis for the null space of the matrix $A = \begin{pmatrix} 1 & -1 & 1 & 2 \\ 2 & 1 & 3 & 2 \\ 1 & 5 & 3 & -2 \end{pmatrix}$ and determine its dimension?

Answer: $\begin{pmatrix} -4/3 \\ -1/3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4/3 \\ 2/3 \\ 0 \\ 1 \end{pmatrix}$ and the dimension of null space of A is 2.

10. Theorem: $A \in \mathbb{M}_{m\times n}(\mathbb{R})$. **Then** $Ker(A) = \{\mathbf{0}\} \iff rank(A) = n$.

For a square matrix the statement is true when A is invertible
(**cf.** remark under point 7 above: When A is invertible, $rref(A) = I_n \implies$ no. of pivots = n = rank(A) by def. Further, $A\mathbf{x} = 0$ can be solved by considering the augmented matrix $rref(A \mid \mathbf{0}) = (I_n \mid \mathbf{0})$) which gives us $x_1 = 0, \; x_2 = 0, \; x_3 = 0$. which gives $Ker(A) = \{\mathbf{0}\}$. The converse is obvious.

11. Theorem (*Rank-nullity theorem*): For any $m \times n$ matrix A, the following is known as the *fundamental theorem of linear algebra*:

$$dim(Null(A)) + dim(Im(A)) = n$$

or equivalently,

$$\text{(nullity of } A) + \text{(rank of } A) = n$$