Tutorial 3: Complex integration and application of Cauchy's theorems

Due: March 1 (Friday) before 1 pm in my office G254

1. Evaluate $\oint_C f(z)dz$ where C is the unit circle centered at the origin.
   Consider $f(z)$ as (i) z^2, (ii) $(z-\frac{1}{2})^{-2}$.

2. State Cauchy's integral theorem. Find $\frac{1}{2\pi i}\oint_C \frac{\zeta}{\zeta - z}d\zeta$, where C is the unit circle $|\zeta| = 1$.

3. State Liouville's theorem. Apply this theorem to prove the *fundamental theorem of algebra* (any polynomial $P(z) = a + 0 + a_1 z + ... + a_m z^m, \quad a_m \neq 0, m \geq 1$ integer, has at least one root).

4. Use Cauchy's integral theorem to show that the value of an analytic function at any interior point in a region bounded by a circle is the mean value of the function integrated over the circle centered at z. Further, show that the value of the function at any interior point equals the mean value over the area of a circle centered at z.

5. Let $C = \{Re^{it} : 0 \leq t \leq \pi, R \in \mathbb{R}\}$ be an open upper semicircle of radius R with its center at the origin. Consider $f(z) = \frac{1}{z^2+a^2}$ and $\int_c f(z)dz$ where $a \in \mathbb{R}$ and $R > a > 0$. Show that

   $$|f(z)| \leq \frac{1}{R^2 - a^2} \implies \left|\int_C f(z)dz\right| \leq \frac{\pi R}{R^2 - a^2}.$$

   Further, find the limit $\int_C f(z)dz$ as $R \to \infty$.

6. Show that $I_R = \int_{C_R} \frac{e^{iz}}{z^2}dz \to 0$ as $R \to \infty$.

7. Consider the integral

   $$I = \int_{-\infty}^{\infty} \frac{1}{x^2+1}dx.$$

   Evaluate the above integral by considering

   $$\oint_C \frac{1}{z^2+1}dz$$

   where $C = C_1 + C_R, C_1$ is the line joining $-R$ and R, and $C_R = \{Re^{it}$ where $t : 0 \to \pi\}$. In other words, C is the closed semicircle in the upper-half $z-$plane with endpoints at $z = -R$ and $z = R$ plus the $x-$axis. Then, verify your answer by usual integration in real variables.

8. Evaluate the integral

   $$\int_{-\infty}^{\infty} \frac{1}{(x+i)^2}dx$$

   by considering $\oint_{C(R)} \frac{1}{(z+i)^2}dz$, where $C_{(R)}$ is the closed semicircle in the upper half plane with corners $z = -R$ and $z = R$, plus the x-axis.

9. Show that the integral $\oint_C \frac{1}{z^2}dz$, where C is a path beginning at $z = -a$ and ending at $z = b, \quad a, b > 0$, is independent of the path as long as C does not go through the origin. Explain why the real valued integral $\int_{-a}^{b} \frac{1}{x^2}dx$ does not exist but the value obtained by formal substitution of limits agrees with the complex integral above.

10. Use Cauchy's theorem to compute $\int_0^{2\pi} \cos^{2p} t dt$. Then use your result to show that

    $$\lim_{p\to\infty} \frac{{}^{2p}C_p}{2^{2p}} = 0.$$