Composite numerical integration

Example Use Simpson's rule to approximate $\int_0^4 e^x\,dx$ and compare this to the results obtained by adding the Simpson's rule approximations for $\int_0^2 e^x\,dx$ and $\int_2^4 e^x\,dx$. Compare these approximations to the sum of Simpson's rule for $\int_0^1 e^x\,dx$, $\int_1^2 e^x\,dx$, $\int_2^3 e^x\,dx$, and $\int_3^4 e^x\,dx$.

Solution Simpson's rule on $[0,4]$ uses $h=2$ and gives

$$\int_0^4 e^x\,dx \approx \frac{2}{3}(e^0+4e^2+e^4)=56.76958.$$

The exact answer in this case is $e^4-e^0=53.59815$, and the error -3.17143 is far larger than we would normally accept.

Applying Simpson's rule on each of the intervals $[0,2]$ and $[2,4]$ uses $h=1$ and gives

$$\begin{aligned}\int_0^4 e^x\,dx &= \int_0^2 e^x\,dx + \int_2^4 e^x\,dx\\ &\approx \frac{1}{3}\left(e^0+4e+e^2\right)+\frac{1}{3}\left(e^2+4e^3+e^4\right)\\ &= \frac{1}{3}\left(e^0+4e+2e^2+4e^3+e^4\right)\\ &= 53.86385.\end{aligned}$$

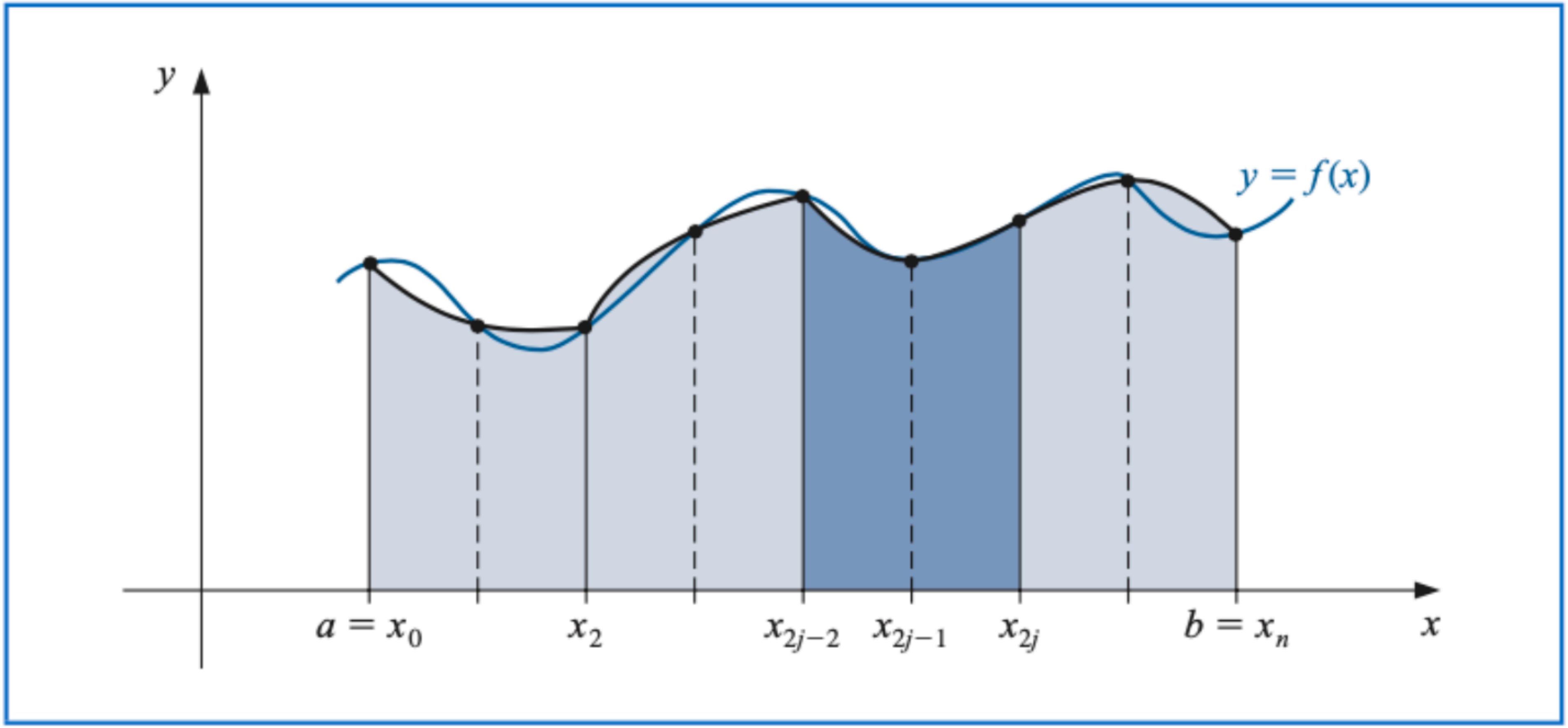
The error has been reduced to -0.26570.

For the integrals on $[0,1]$,$[1,2]$,$[3,4]$, and $[3,4]$ we use Simpson's rule four times with $h=\frac{1}{2}$ giving

$$\begin{aligned}\int_0^4 e^x\,dx &= \int_0^1 e^x\,dx + \int_1^2 e^x\,dx + \int_2^3 e^x\,dx + \int_3^4 e^x\,dx\\ &\approx \frac{1}{6}\left(e_0+4e^{1/2}+e\right)+\frac{1}{6}\left(e+4e^{3/2}+e^2\right)\\ &\quad+\frac{1}{6}\left(e^2+4e^{5/2}+e^3\right)+\frac{1}{6}\left(e^3+4e^{7/2}+e^4\right)\\ &= \frac{1}{6}\left(e^0+4e^{1/2}+2e+4e^{3/2}+2e^2+4e^{5/2}+2e^3+4e^{7/2}+e^4\right)\\ &= 53.61622.\end{aligned}$$

The error for this approximation has been reduced to -0.01807. ■

To generalize this procedure for an arbitrary integral $\int_a^b f(x)\,dx$, choose an even integer n. Subdivide the interval $[a, b]$ into n subintervals, and apply Simpson's rule on each consecutive pair of subintervals.



With $h = (b - a)/n$ and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$, we have

$$\int_a^b f(x)\,dx = \sum_{j=1}^{n/2} \int_{x_{2j-2}}^{x_{2j}} f(x)\,dx$$

$$= \sum_{j=1}^{n/2} \left\{ \frac{h}{3}[f(x_{2j-2}) + 4f(x_{2j-1}) + f(x_{2j})] - \frac{h^5}{90} f^{(4)}(\xi_j) \right\},$$

for some ξ_j with $x_{2j-2} < \xi_j < x_{2j}$, provided that $f \in C^4[a, b]$.

Theorem Let $f \in C^4[a,b]$, n be even, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a,b)$ for which the **Composite Simpson's rule** for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{3}\left[f(a) + 2\sum_{j=1}^{(n/2)-1} f(x_{2j}) + 4\sum_{j=1}^{n/2} f(x_{2j-1}) + f(b)\right] - \frac{b-a}{180}h^4 f^{(4)}(\mu).$$

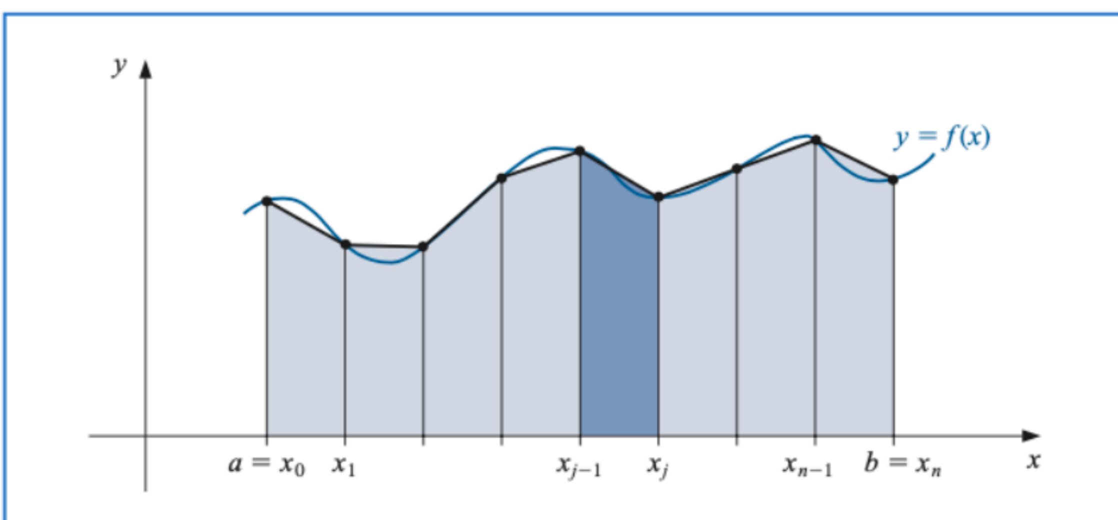
■

Notice that the error term for the Composite Simpson's rule is $O(h^4)$, whereas it was $O(h^5)$ for the standard Simpson's rule. However, these rates are not comparable because for standard Simpson's rule we have h fixed at $h = (b-a)/2$, but for Composite Simpson's rule we have $h = (b-a)/n$, for n an even integer. This permits us to considerably reduce the value of h when the Composite Simpson's rule is used.

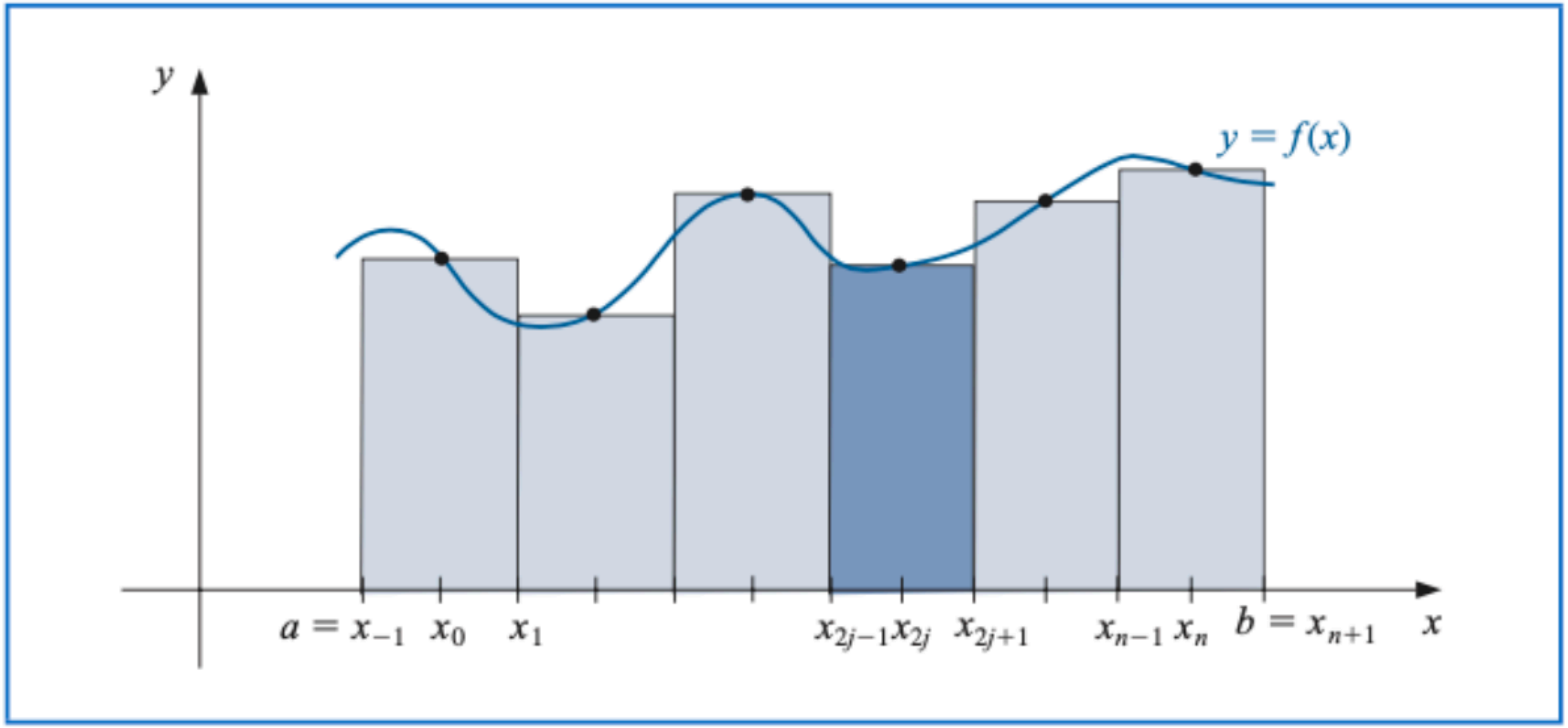
Theorem Let $f \in C^2[a,b]$, $h = (b-a)/n$, and $x_j = a + jh$, for each $j = 0, 1, \ldots, n$. There exists a $\mu \in (a,b)$ for which the **Composite Trapezoidal rule** for n subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{b-a}{12}h^2 f''(\mu).$$

Figure



Figure



Theorem Let $f \in C^2[a,b]$, n be even, $h = (b-a)/(n+2)$, and $x_j = a + (j+1)h$ for each $j = -1, 0, \ldots, n+1$. There exists a $\mu \in (a,b)$ for which the **Composite Midpoint rule** for $n+2$ subintervals can be written with its error term as

$$\int_a^b f(x)\,dx = 2h\sum_{j=0}^{n/2} f(x_{2j}) + \frac{b-a}{6}h^2 f''(\mu).$$

Example Determine values of h that will ensure an approximation error of less than 0.00002 when approximating $\int_0^\pi \sin x \, dx$ and employing
(a) Composite Trapezoidal rule and **(b)** Composite Simpson's rule.

Solution **(a)** The error form for the Composite Trapezoidal rule for $f(x) = \sin x$ on $[0, \pi]$ is

$$\left| \frac{\pi h^2}{12} f''(\mu) \right| = \left| \frac{\pi h^2}{12} (-\sin \mu) \right| = \frac{\pi h^2}{12} |\sin \mu|.$$

To ensure sufficient accuracy with this technique we need to have

$$\frac{\pi h^2}{12}|\sin \mu| \leq \frac{\pi h^2}{12} < 0.00002.$$

Since $h = \pi/n$ implies that $n = \pi/h$, we need

$$\frac{\pi^3}{12n^2} < 0.00002 \quad \text{which implies that} \quad n > \left(\frac{\pi^3}{12(0.00002)}\right)^{1/2} \approx 359.44.$$

and the Composite Trapezoidal rule requires $n \geq 360$.

(b) The error form for the Composite Simpson's rule for $f(x) = \sin x$ on $[0, \pi]$ is

$$\left|\frac{\pi h^4}{180} f^{(4)}(\mu)\right| = \left|\frac{\pi h^4}{180} \sin \mu\right| = \frac{\pi h^4}{180} |\sin \mu|.$$

To ensure sufficient accuracy with this technique we need to have

$$\frac{\pi h^4}{180} |\sin \mu| \le \frac{\pi h^4}{180} < 0.00002.$$

Using again the fact that $n = \pi/h$ gives

$$\frac{\pi^5}{180 n^4} < 0.00002 \quad \text{which implies that} \quad n > \left(\frac{\pi^5}{180(0.00002)}\right)^{1/4} \approx 17.07.$$

So Composite Simpson's rule requires only $n \ge 18$.

Composite Simpson's rule with $n = 18$ gives

$$\int_0^{\pi} \sin x \, dx \approx \frac{\pi}{54}\left[2\sum_{j=1}^{8} \sin\left(\frac{j\pi}{9}\right) + 4\sum_{j=1}^{9} \sin\left(\frac{(2j-1)\pi}{18}\right)\right] = 2.0000104.$$

This is accurate to within about 10^{-5} because the true value is $-\cos(\pi) - (-\cos(0)) = 2$.