Application of law of total probability and law of total expectation: Random Walk

Amrik Sen

PCL 108: Statistical Methods & Algorithms
Thapar Institute of Engineering & technology, Patiala

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Definitions, Theorem's - recap

1. Law of total probability: $P(A) = \sum_n P(A|B_n)P(B_n)$.
2. Law of total expectation: $E(A) = \sum_n E(A|B_n)P(B_n)$.

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Ignore following theorems for now!

3. **Thm.** If p_{ij} is **doubly stochastic** probability transition matrix (& $S = \{1, 2, ...N\}$ is a finite state space), then *there exists* a **stationary distribution** $\Pi(x) = \frac{1}{N}$ for all $x \in S$.

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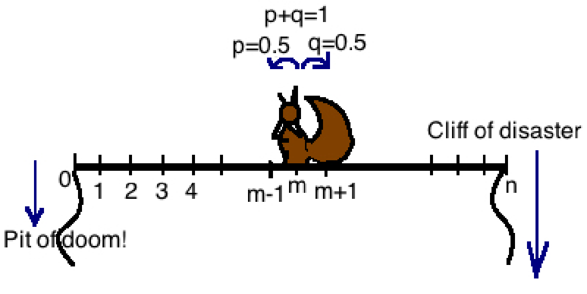
4. **Thm.** If p_{ij} is **irreducible** and has a stationary distribution Π, then $\Pi(x) = \frac{1}{E_x T_x}$.

   $T_x = \min\{n > 0$ such that $X_n = x\}$ (1^{st} **return time to** x).

Squirrel out for a random walk on an island



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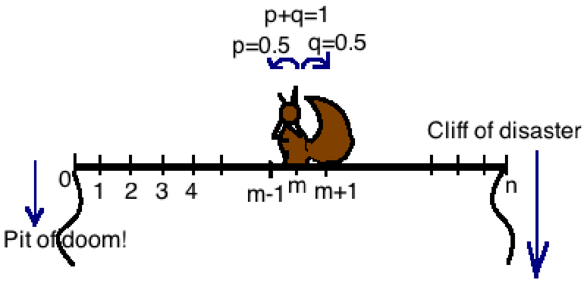

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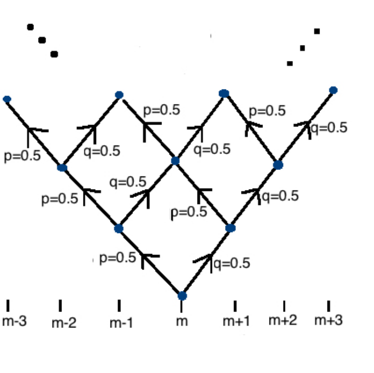
2) What's the squirrel's life expectancy in terms of no. of hops? Does his initial position change his chances of surviving longer?

Tree Diagram



Guess: since the tree spans the entire state space (including the boundaries), perhaps there is no escape for the squirrel!

Probability that the squirrel will die ..

W: event that the squirrel falls in to the pit (left). We want to compute:

$$P_m = P_m(\text{left pit}) = \text{Probability of } W \text{ when he starts at } X_0 = m, \qquad (1)$$

with $P_0 = 1, \;\; P_n = 0.$

Let E be the event that the first hop is to the left. We will **condition** ⚠ our computation upon this event E as follows:

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Let E be the event that the first hop is to the left. We will **condition** ⚠ our computation upon this event E as follows:

$$\begin{aligned}
P_m &= P(W \text{ and } E | X_0 = m) + P(W \text{ and } \overline{E} | X_0 = m) \\
&= P(W | E \wedge X_0 = m) P(E | X_0 = m) + P(W | \overline{E} \wedge X_0 = m) P(\overline{E} | X_0 = m) \\
&= P(W | X_1 = m-1) \times \frac{1}{2} + P(W | X_1 = m+1) \times \frac{1}{2} \\
&\overset{\text{indep. hops}}{=} \frac{1}{2} P(W | X_0 = m-1) + \frac{1}{2} P(W | X_0 = m+1) = \boxed{\frac{1}{2} P_{m-1} + \frac{1}{2} P_{m+1}}
\end{aligned}$$

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Symmetrical solution

What is the probability that starting from the same initial position, he falls off the cliff on the right at $x = n$? i.e. $P_m(\text{right cliff}) = ?$

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Now check that $P_m(\text{left pit}) + P_m(\text{right cliff}) = 1$, i.e. **the squirrel will eventually fall off the edge and die**!

Life expectancy?

Let D be the number of hops (steps) before he falls off the edge. We will use the **law of total expectation** and once again **condition** ⚠ upon the event E as follows:

$$
\begin{aligned}
E_m &= E(D|X_0 = m) \\
&= E(D|E \wedge X_0 = m)P(E|X_0 = m) + E(D|\overline{E} \wedge X_0 = m)P(\overline{E}|X_0 = m) \\
&= \frac{1}{2}E(D|X_1 = m-1) + \frac{1}{2}E(D|X_1 = m+1) \\
&\overset{\text{reset chain}}{=} \frac{1}{2}\left\{1 + E(D|X_0 = m-1)\right\} + \frac{1}{2}\left\{1 + E(D|X_0 = m+1)\right\} \\
&= 1 + \frac{1}{2}E_{m-1} + \frac{1}{2}E_{m+1}
\end{aligned}
$$

Life expectancy?

Again we have a recurrence relation $E_{m+1} - 2E_m + E_{m-1} = -2$, we use the roots of the characteristic equation $r^2 - 2r + 1 = 0$ along with $E_0 = E_n = 0$ to find $E_m = m(n - m)$, i.e. his life expectancy is the product of his distances from the two edges.

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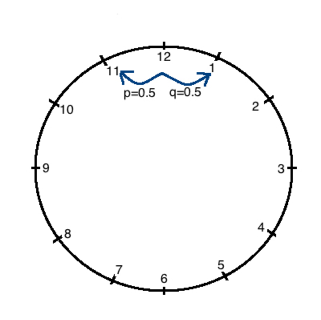
Where should he start from to have a larger life span?
Obtain the maximum of the function $f(m) = m(n-m)$ ... $m = n/2$!
⚠ Careful with discrete space if you are planning to employ calculus machinery!

Random Walk on a Ring; $\{X_n\}$ is a Markov chain



1) What is the expected no. of steps that X_n will take before returning to its starting position?

2) What is the probability that X_n will visit all other states before returning to its starting position?

p is **doubly stochastic, irreducible**

$$
p \;=\;
\begin{array}{c}
\\ 1 \\ 2 \\ 3 \\ 4 \\ \cdot \\ \cdot \\ \cdot \\ 11 \\ 12
\end{array}
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & \cdot & \cdot & \cdot & \cdot & 11 & 12 \\
\end{array}
$$

$$
p = \begin{array}{c|cccccccccc}
 & 1 & 2 & 3 & 4 & \cdot & \cdot & \cdot & \cdot & 11 & 12 \\
\hline
1 & 0 & 1/2 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 0 & 1/2 \\
2 & 1/2 & 0 & 1/2 & 0 & \cdot & & & & 0 & 0 \\
3 & 0 & 1/2 & 0 & 1/2 & 0 & & & & 0 & 0 \\
4 & 0 & 0 & 1/2 & 0 & 1/2 & \cdot & \cdot & \cdot & 0 & 0 \\
\cdot & \cdot & & & \cdot & & \cdot & & & \cdot & \cdot \\
\cdot & \cdot & & & & \cdot & & & \cdot & 0 & \cdot \\
\cdot & \cdot & & & & & \cdot & & & 1/2 & \cdot \\
11 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 1/2 & 0 & 1/2 \\
12 & 1/2 & 0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & 1/2 & 0
\end{array}
$$

Therefore, $\exists$ a stationary distribution $\Pi(x) = \frac{1}{12} \quad \forall x \in \{1, 2, 3 \ldots 12\}$; and, $E_x T_x = \frac{1}{\Pi(x)} = 12$; $T_x = min\{n > 0 \text{ s.t. } X_n = x\}$ is first return time to x.

Probability of visiting all other states before returning to start, ϕ

WLOG, we consider $x = 12$(or equivalently 0) to be that starting point and make the first move to $x = 1$.
Like in the case of the random walk on a line, we will condition ⚠ upon the first move to $x = 1$.

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Let $\phi(m) = Prob$(we reach 11 before hitting 12 starting from m).
We want to find $\phi(1)$**!** In this set up, $\phi(12) = \phi(0) = 0$ and $\phi(11) = 1$.

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$$\phi(m) = \sum_{n \in S} p(m,n)\phi(n),$$

which leads to a recurrence relation $\phi_m = \frac{1}{2}\phi_{m-1} + \frac{1}{2}\phi_{m+1}$.

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The solution is $\phi_m = m/11$ that gives us $\boxed{\phi(1) = \frac{1}{11}}$.

Applications of random walk in science and engineering

- Brownian motion is the limit of symmetric random walk (take infinitesimally smaller step sizes).
- Molecular motion in a fluid.
- Price of a fluctuating stock in the financial market.
- (Neuroscience): modeling neurons firing in the brain.
- Network dynamics in wireless networks.
- Population dynamics.
- Quantum field theory.
- Polymer science.
- Check out the sculpture **Quantum Cloud** in London (made using a random walk model)!