Chapter 2

Module 2, Lecture 4

2.1 Agenda Items

- *Eigenvalues (evs) and Eigenvectors (EVs) of a matrix.*
- *Meaning of evs and EVs.*
- *Diagonalizable matrices and similar transformations.*
- *Analytical (pen-paper) method of finding evs.*
- *computational method of finding evs of a matrix(power method, etc).*

Definition 8 (evs and EVs)**.** Let $A \in \mathbf{M}_{n\times n}(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A nonzero vector $x \in \mathbb{F}^n$ is an EV of A if $Ax = \lambda x$ for some $\lambda \in \mathbb{F}$. λ is said to be an ev A corresponding to the EV x.

2.2 Meaning of the equation $AX = \lambda x$

2.2.1 Algebraic meaning

$Ax = \lambda x$ can also be written as $(A-\lambda I)x = 0$, i.e., $\ker(A-\lambda I) = EVs \sqcup \{0\}$*. In the above equation we use that* $\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$*. The subspace* $\ker(A-\lambda I)$ *has a special name,* EIGENSPACE of λ w.r.t. A.

Consider an example: $A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$ so that $A - \lambda I = \begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}$. Now solving $Ax = \lambda x$ is equivalent to solving the system of linear equations $\begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This implies that

$$
\begin{aligned}
(2-\lambda)x_1 - x_2 &= 0 \\
2x_1 + (4-\lambda)x_2 &= 0
\end{aligned}
$$

Since EV cannot be 0, finding Evs of A boils down to the following question:

When does this system of linear equations have a nontrivial solution?

To answer the above question we need to know various features of an invertible matrices:

Let $B \in \mathbf{M}_{n\times n}(\mathbb{F})$, where $F = \mathbb{R}$ or $\mathbb{C}$. TFAE

- B is invertible.
- $Bx = b$ has a unique solution in $\mathbb{F}^n$ for all $b \in \mathbb{F}^n$.
- $\text{rref}(B) = I_n$.
- $\textbf{rank}(B) = n$.
- $\text{im}(B) = \mathbb{F}^n$.
- $\ker(B) = \{0\}$.

Let us try to answer the above question now. In view of the above equivalence

$$
\begin{aligned}
\ker(A - \lambda I) \neq \{0\} &\iff (A - \lambda I) \text{ is not invertible} \\
&\iff \det(A - \lambda I) = 0.
\end{aligned}
$$

In our problem

$$
\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) + 2 = \lambda^2 - 6\lambda + 10
$$

The above polynomial in λ is called the *characteristic polynomial for the matrix A. Thus*

$$\det(A - \lambda I) = 0 \iff \lambda = 3 \pm i$$

Let us call $\lambda_1 = 3 + i$ *and* $\lambda_2 = 3 - i$.

To find EV w.r.t. λ_1 *solve* $Ax = \lambda_1 x$. *After solving we obtain* $(1+i)x_1 + x_2 = 0$, *i.e.,* $x_2 = -(1+i)x_1$. *We can take* x_1 *to be any nonzero scalar of* $\mathbb{F}$, *say* k, *so as to write* $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k \\ -(1+i)k \end{pmatrix} = k \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$. *Hence any nonzero multiple of the vector* $\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$ *is an EV of the matrix A w.r.t the ev* λ_1. *Similarly one can find EV corresponding to the ev* λ_2.

2.3 A Slight Digression

Let $B \in \mathbf{M}_{n\times n}(\mathbb{F})$, *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$.

Question: Why null(B) $= \{0\} \iff B$ *is invertible ?*

Answer: For finite dimensional vector spaces U, V *over* $\mathbb{F}$, *a linear transformation* $T : U \to V$ *is invertible if and only if* T *is one to one and onto.*
Rank Nullity Theorem:

$$\textit{nullity}(T) + \mathbf{rank}(T) = \dim(U).$$

Since T *is one to one* $\ker(T) = \{0\}$, *i.e., nullity*$(T) = 0$. *Also since* T *is onto,* $\mathbf{rank}(T) = \dim(V)$. *Therefore by Rank nullity theorem we obtain*

$$T \textit{ is an isomorphism} \implies \dim(U) = \dim(V).$$

2.4 HW/Exercise problem

Q. Consider $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

1. *Find the characteristic polynomial for A.*

2. *Find the evs of A.*

3. *Find the EVs of A*

Ans:

$$\text{evs: } -1, 2, 3.$$
$$\text{EVs: } \begin{pmatrix}1\\1\\2\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Geometrical meaning of $Ax = \lambda x$, when λ is real. Ax is parallel to x, i.e., the "EV" x either gets stretched longitudinally when acted upon by the matrix A.

2.5 Coming Soon!

- *Diagonalizable matrix*
- *Similarity transformation*
- *Application of evs and EVs in solution to ODE*

Chapter 3

Module 2, Lecture 5

3.1 Agenda Item

- *Diagonalization of matrices*
- *Similarity transformation*
- *Spectral decomposition of matrices*

Last Lecture:

- *We define evs and EVs of a square matrix*
- *determinant and trace of a matrix and its relation with evs*

3.2 Diagonalizable Matrices

Certain forms of matrices are convenient to work with. For example

- *Upper/Lower triangular matrices*(*why?*)
- *Diagonal forms*(*why?*)

Think finding evs and powers of above matrices.

Wouldn't it be nice if

$$A \longrightarrow D$$
$$(\textit{any } n \times n \textit{ matrix}) \quad (\textit{diagonal form})$$

$A \in \mathbf{M}_{n\times n}(\mathbb{F})$ is diagonalizable over $\mathbb{F}$ if there exists an invertible matrix S over $\mathbb{F}$ such that $A = SDS^{-1}$, or equivalently $D = S^{-1}AS$.

Note that the evs of A and D will be the same and the above relation $D = S^{-1}AS$ is known as the similarity transformation.

Q. When is a matrix diagonalizable?

Ans: $A \in \mathbf{M}_{n\times n}(\mathbb{F})$ is diagonalizable if and only if A has n linearly independent EVs in $\mathbb{F}^n$.

Note that an $n \times n$ complex matrix that has n distinct eigenvalues is diagonalizable.

Example 9. Q. Find a matrix that diagonalizes $A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$.

Ans: Solve $\det(A - \lambda I) = 0$ to obtain $\lambda_1 = 3 + i$ and $\lambda_2 = 3 - i$. Solving $Ax = \lambda_i x$ for $i = 1, 2$, we obtain

$$X_1 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

as Evs of A w.r.t. the evs λ_1, λ_2, respectively. We note that $S = \begin{pmatrix} 1 & 1 \\ -1-i & -1+i \end{pmatrix}$ diagonalizes A. Since

$$\begin{aligned} S^{-1}AS &= \begin{pmatrix} \frac{-1+i}{2i} & -\frac{1}{2i} \\ \frac{1+i}{2i} & \frac{1}{2i} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1-i & -1+i \end{pmatrix} \\ &= \begin{pmatrix} 3+i & 0 \\ 0 & 3-i \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ &= D \end{aligned}$$

The Column vectors of S form an eigenbasis for A and the diagonal entries of D are the associated evs.

Q. What are the evs and EVs of the $n \times n$ identity matrix I_n?

Is there an eigenbasis for I_n?

Which matrix diagonalizes I_n?

This is in some sense a silly and yet a conceptually trick question.

Example 10. Find the eigenspace of $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

The evs are given by 0 and 1 with algebraic multiplicity 1 and 2, respectively. To find EV consider

$$\begin{aligned} X_1 &= \ker(A - 1I) \\ &= \ker \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \ker \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathrm{sp} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is the reduced row echelon form of the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The calculation:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} x_2 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Above calculation shows that $x_2 = x_3 = 0$. Thus we can take any nonzero value as x_1 to obtain an EV of A w.r.t. the ev 1. For convenience we take $x_1 = 1$ to obtain $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ as an EV. Likewise $X_2 = \ker A = \text{sp}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Thus we are able to find only two linearly independent EVs. Hence we won't have an eigenbasis here, equivalently we cannot find S to diagonalize A.

3.3 Geometric multiplicity of ev

$$\begin{aligned} gemm(\lambda) &= \dim(\ker(A - \lambda I_n)) \\ &= nullity(A - \lambda I_n) \\ &= n - \mathbf{rank}(A - \lambda I_n) \end{aligned}$$

In previous example

$$gemm(1) = \dim(\ker(A - \lambda I_n)) = \dim\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle = 1 \neq almu(1) = 2.$$

Theorem 11. *A matrix A is orthogonally diagonalizable ($D = Q^{-1}AQ \equiv Q^tAQ$) iff A is symmetric ($A = A^t$).*

3.4 Spectral decomposition

Let A be a real symmetric $n \times n$ matrix with evs $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding orthonormal EVs $v_1, v_2 \ldots, v_n$; then

$$\begin{aligned} A &= \begin{pmatrix} \vdots & \vdots & \vdots \\ v_1 & v_2 & v_3 \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \lambda_1 & & \mathit{0} \\ & \ddots & \\ \mathit{0} & & \lambda_n \end{pmatrix} \begin{pmatrix} \cdots & v_1 & \cdots \\ \cdots & v_2 & \cdots \\ & \vdots & \\ \cdots & v_n & \cdots \end{pmatrix} \\ &= QDQ^t. \end{aligned}$$

This concludes the life and theory of a matrix in FM112.