19/9/22 Application of null sp. of a matrix: Bank transaction matrix model.

* Recall, we agreed that in a closed accounting system w/ n accounts; the most canonical transactions may be represented by the following set of vectors:

$$\begin{pmatrix}1\\-1\\0\\0\\\vdots\\0\end{pmatrix};\ \begin{pmatrix}1\\0\\-1\\0\\0\\\vdots\\0\end{pmatrix};\ \cdots\cdots;\ \begin{pmatrix}1\\0\\0\\\vdots\\0\\-1\end{pmatrix}$$

* Further, we agreed that at any instance, if $t_1, t_2, \ldots, t_n$ denote the "net" transactions happening w.r.t. the "n" respective accounts, then $t_1 + t_2 + \cdots + t_n = 0$ — (1) b/c of the fact that the accounting system is closed.

* Now can we find an appropriate vector space for the transactions?

Consider the matrix $A = \begin{pmatrix}1 & 1 & 1 & \cdots & 1\\ 0 & 0 & \cdots & & 0\\ 0 & \vdots & & \ddots & \vdots\\ \vdots & & & & \\ 0 & 0 & \cdots & \cdots & 0\end{pmatrix}$ — (2)

& vector $\begin{pmatrix}t_1\\t_2\\\vdots\\t_n\end{pmatrix}$ clearly $A\vec{t} = \vec{0}$ is the same as eq (1).

So, clearly any transaction represented by the transaction vector $\begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{pmatrix}$ is in the null space of the matrix A. Hence, the transaction space ~~of A~~ is the null sp. of A which is a vector space.

* Infact; had we chosen the matrix A to be $\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}$

or ... the idea above would still work but the matrix A given by expression (2) is the rref form, so we will chose this matrix.

* Now let's calculate the basis of the transaction space $\mathbb{T} \equiv \text{null}(A)$.

Since $a_{11} = 1$ is the pivot; we define $t_2 = c_2,\ t_3 = c_3, \ldots, t_n = c_n$ arbitrarily. & express $t_1 = -c_2 - c_3 - \cdots - c_n$ in the soln. $\vec{t}$ for $A\vec{t} = 0$; i.e. $\vec{t} = \begin{pmatrix} -c_2 - c_3 - \cdots - c_n \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix}$

Next we align the free parameters (c_is) appropriately to enable us to write:

$$\vec{t} = c_2\begin{pmatrix}-1\\1\\0\\\vdots\\0\end{pmatrix} + c_3\begin{pmatrix}-1\\0\\1\\0\\\vdots\\0\end{pmatrix} + \cdots + c_n\begin{pmatrix}-1\\0\\\vdots\\0\\1\end{pmatrix}$$

By inspection, we see that these basis vectors of Π were indeed the canonical transactions that we had identified at the very beginning. No wonder!!

So the basis vectors are

$$\left.\begin{aligned}&\hat{t}_2 = \begin{pmatrix}-1\\1\\0\\\vdots\\0\end{pmatrix};\ \hat{t}_3 = \begin{pmatrix}-1\\0\\1\\0\\\vdots\\0\end{pmatrix};\ \cdots;\ \hat{t}_n = \begin{pmatrix}-1\\0\\\vdots\\0\\1\end{pmatrix}\\ &\hat{t} = \{\hat{t}_2, \hat{t}_3, \cdots, \hat{t}_n\}\\ &\dim(\hat{t}) = (n-1) = \dim(\Pi)\end{aligned}\right\} \quad (3)$$

* Now let us say, that we have a frozen account among the n accounts in this bank. Let's say for simplicity, $n=5$ & the 4^{th} account is frozen.

A frozen account means: $t_4 = 0$ always.
(n=4)

so $A = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ — (A)

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$A\vec{t} = \vec{0}$ as before

So all transactions $\vec{t} \in \text{null}(A)$

In order to find the basis of above Π; we again set $t_2 = c_2, t_3 = c_3, t_4 = 0, t_5 = c_5$

Whence $t_1 = -c_2 - c_3 - c_5$

so $\hat{t}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\hat{t}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$; $\hat{t}_5 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$; $\hat{t}_4 = \vec{0}$

$\therefore$ the basis set $\hat{t} = \{\hat{t}_2, \hat{t}_3, \hat{t}_5\}$

$\dim(\Pi) = 3$.

* If we have tandem accounts:

$\underbrace{t_1 \longleftrightarrow t_3}_{\Pi_1}$ and $\underbrace{t_2 \longleftrightarrow t_4,\ t_2 \longleftrightarrow t_5,\ t_4 \longleftrightarrow t_5}_{\Pi_2}$ but Π_1 & Π_2 do <u>not</u> transact.

Then $A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ & $\Pi = \text{null}(A)$

Basis: $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}; \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\therefore \dim(\Pi) = 3$

Further $\mathbb{T} = \mathbb{T}_1 \oplus \mathbb{T}_2$ — (5)

where $\mathbb{T}_1$ & $\mathbb{T}_2$ are <u>orthogonal</u> <u>subspaces</u> of $\mathbb{T}$;

i.e. for any $r_1 \in \mathbb{T}_1$ and any $r_2 \in \mathbb{T}_2$

$$\langle r_1, r_2 \rangle = 0 \quad \text{— (6)}$$

i.e. $r_1 \perp r_2$.

this is easy to check

$$\left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle = 0 \ \& \ \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = 0.$$

So clearly (6) will also be true.

* Now let us say we "open" the transactions/accounting system in such a way that the account (5) serves as the gateway to external transactions: $t_1 + t_2 + t_3 + t_4 + t_5 = \alpha$

Consider $A = \begin{pmatrix} -1 & -1 & -1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$

Let $\mathbb{T} = \mathrm{Col}(A) \equiv \mathrm{Im}(A)$.

$B = \mathrm{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

$\therefore$ $\mathrm{Col}(B)$ is isomorphic to $\mathrm{Col}(A)$

We have $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

as the basis of $\mathbb{T}$ as was expected: $\dim(\mathbb{T}) = 5$.

$$\vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix} = t_1 \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \dots + t_5 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -t_1 \\ t_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -t_2 \\ 0 \\ t_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -t_3 \\ 0 \\ 0 \\ t_3 \\ 0 \end{pmatrix} + \begin{pmatrix} -t_4 \\ 0 \\ 0 \\ 0 \\ t_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ t_5 \end{pmatrix}$$

Sum of Components $= 0$

Sum of Component $= t_5 = \alpha$

** Now this should give you a cue that in the very first model of the closed acc. sys we might have chosen $\Pi = \text{Col}(\tilde{A})$ where $\tilde{A} = \begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Does this ensure closure? $\text{rref}(\tilde{A}) = ?$