Gaussian Elimination with Partial Pivoting

In the problem below, we have order of magnitude differences between coefficients in the different rows.

$$\left[\begin{array}{cccc|c} 0.02 & 0.01 & 0 & 0 & 0.02 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

Step 0a: Find the entry in the left column with the largest absolute value. This entry is called the pivot.

Step 0b: Perform row interchange (if necessary), so that the pivot is in the first row.

$$\left[\begin{array}{cccc|c} 0.02 & 0.01 & 0 & 0 & 0.02 \\ \textcircled{1} & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} \textcircled{1} & 2 & 1 & 0 & 1 \\ 0.02 & 0.01 & 0 & 0 & 0.02 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

Step 1: Gaussian Elimination

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0.02 & 0.01 & 0 & 0 & 0.02 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

Step 2: Find new pivot

Step 3: Switch rows (if necessary)

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & \textcircled{1} & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

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$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & \textcircled{1} & 2 & 1 & 4 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

Step 4: Gaussian Elimination

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & -0.03 & -0.02 & 0 & 0 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0.04 & 0.03 & 0.12 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

Step 5: Find new pivot

Step 6: Switch rows (if necessary)

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 0.04 & 0.03 & 0.12 \\ 0 & 0 & 100 & 200 & 800 \end{array}\right]$$

$$\downarrow$$

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Step 7: Gaussian Elimination

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & \boxed{100} & \boxed{200} & \boxed{800} \\ 0 & 0 & 0.04 & 0.03 & 0.12 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 4 \\ 0 & 0 & 100 & 200 & 800 \\ 0 & 0 & 0 & -0.05 & -0.2 \end{array}\right]$$

Step 8: Back Substitute

$$-0.2x_4 = -0.05; \quad \mathbf{x_4 = 4}$$

$$100x_3 + 200x_4 = 800; \quad \mathbf{x_3 = 0}$$

$$x_2 + 2x_3 + x_4 = 4; \quad \mathbf{x_2 = 0}$$

$$x_1 + 2x_2 + x_3 = 1; \quad \mathbf{x_1 = 1}$$

Pivoting helps reduce rounding errors; you are less likely to add/subtract with very small number (or very large) numbers.

Gaussian Elimination with Partial Pivoting

Can Partial Pivoting fail?

- Each multiplier m_{ji} in the partial pivoting algorithm has magnitude less than or equal to 1.
- Although this strategy is sufficient for many linear systems, situations do arise when it is inadequate.
- The following (contrived) example illusrates the point.

Gaussian Elimination with Partial Pivoting

Example: When Partial Pivoting Fails

The linear system

$$\begin{aligned} E_1: &\quad 30.00x_1 + 591400x_2 = 591700 \\ E_2: &\quad 5.291x_1 - \quad 6.130x_2 = 46.78 \end{aligned}$$

is the same as that in the two previous examples except that all the entries in the first equation have been multiplied by 10^4.

Gaussian Elimination with Partial Pivoting

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The partial pivoting procedure described in the algorithm with 4-digit arithmetic leads to the same incorrect results as obtained in the first example (Gaussian elimination without pivoting).

Gaussian Elimination with Partial Pivoting

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Apply Partial Pivoting

Gaussian Elimination with Partial Pivoting

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Apply Partial Pivoting

The maximal value in the first column is 30.00, and the multiplier

$$m_{21} = \frac{5.291}{30.00} = 0.1764$$

Gaussian Elimination with Partial Pivoting

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leads to the system

$$\begin{aligned} 30.00x_1 + 591400x_2 &\approx 591700 \\ -104300x_2 &\approx -104400 \end{aligned}$$

Gaussian Elimination with Partial Pivoting

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$$\begin{aligned} 30.00x_1 + 591400x_2 &\approx 591700 \\ -104300x_2 &\approx -104400 \end{aligned}$$

which has the same inaccurate solutions as in the first example: $x_2 \approx 1.001$ and $x_1 \approx -10.00$.

Outline

Gaussian Elimination with Scaled Partial Pivoting

Scaled Partial Pivoting

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- Scaled partial pivoting places the element in the pivot position that is largest relative to the entries in its row.

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- Scaled partial pivoting places the element in the pivot position that is largest relative to the entries in its row.
- The first step in this procedure is to define a scale factor s_i for each row as

$$s_i = \max_{1 \leq j \leq n} |a_{ij}|$$

Gaussian Elimination with Scaled Partial Pivoting

Scaled Partial Pivoting

- Scaled partial pivoting places the element in the pivot position that is largest relative to the entries in its row.
- The first step in this procedure is to define a scale factor s_i for each row as
$$s_i = \max_{1 \le j \le n} |a_{ij}|$$
- If we have $s_i = 0$ for some i, then the system has no unique solution since all entries in the ith row are 0.

Gaussian Elimination with Scaled Partial Pivoting

Scaled Partial Pivoting (Cont'd)

- Assuming that this is not the case, the appropriate row interchange to place zeros in the first column is determined by choosing the least integer p with

$$\frac{|a_{p1}|}{s_p} = \max_{1 \le k \le n} \frac{|a_{k1}|}{s_k}$$

and performing $(E_1) \leftrightarrow (E_p)$.

Gaussian Elimination with Scaled Partial Pivoting

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- The effect of scaling is to ensure that the largest element in each row has a relative magnitude of 1 before the comparison for row interchange is performed.

Gaussian Elimination with Scaled Partial Pivoting

Scaled Partial Pivoting (Cont'd)

- In a similar manner, before eliminating the variable x_i using the operations

$$E_k - m_{ki}E_i, \quad \text{for } k = i+1, \ldots, n,$$

we select the smallest integer $p \geq i$ with

$$\frac{|a_{pi}|}{s_p} = \max_{i \leq k \leq n} \frac{|a_{ki}|}{s_k}$$

and perform the row interchange $(E_i) \leftrightarrow (E_p)$ if $i \neq p$.

Gaussian Elimination with Scaled Partial Pivoting

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- The scale factors $s_1, \ldots, s_n$ are computed only once, at the start of the procedure.

Gaussian Elimination with Scaled Partial Pivoting

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- The scale factors $s_1, \ldots, s_n$ are computed only once, at the start of the procedure.
- They are row dependent, so they must also be interchanged when row interchanges are performed.

Gaussian Elimination with Scaled Partial Pivoting

Example

Returning to the previous ewxample, we will appl scaled partial pivoting for the linear system:

$$\begin{aligned} E_1 : &\quad 30.00x_1 + 591400x_2 = 591700 \\ E_2 : &\quad 5.291x_1 - \quad 6.130x_2 = 46.78 \end{aligned}$$

Gaussian Elimination with Scaled Partial Pivoting

$$
\begin{aligned}
E_1: &\quad 30.00x_1 + 591400x_2 = 591700 \\
E_2: &\quad 5.291x_1 - \quad 6.130x_2 = 46.78
\end{aligned}
$$

Solution (1/2)

Gaussian Elimination with Scaled Partial Pivoting

$$\begin{aligned} E_1 &: \quad 30.00x_1 + 591400x_2 = 591700 \\ E_2 &: \quad 5.291x_1 - \quad 6.130x_2 = 46.78 \end{aligned}$$

Solution (1/2)

We compute

$$s_1 = \max\{|30.00|, |591400|\} = 591400$$

and

$$s_2 = \max\{|5.291|, |-6.130|\} = 6.130$$

Gaussian Elimination with Scaled Partial Pivoting

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Solution (1/2)

We compute

$$s_1 = \max\{|30.00|, |591400|\} = 591400$$

and

$$s_2 = \max\{|5.291|, |-6.130|\} = 6.130$$

so that

$$\frac{|a_{11}|}{s_1} = \frac{30.00}{591400} = 0.5073 \times 10^{-4}, \qquad \frac{|a_{21}|}{s_2} = \frac{5.291}{6.130} = 0.8631,$$

and the interchange $(E_1) \leftrightarrow (E_2)$ is made.

Gaussian Elimination with Scaled Partial Pivoting

Solution (2/2)

Applying Gaussian elimination to the new system

$$5.291x_1 - 6.130x_2 = 46.78$$
$$30.00x_1 + 591400x_2 = 591700$$

produces the correct results: $x_1 = 10.00$ and $x_2 = 1.000$.