Solutions to Systems of Linear Differential Equations (DE)

An n-dimensional **linear first-order DE system** is one that can be written as a matrix vector equation -

$$\vec{X}'(t) = A(t)\vec{X}(t) + \vec{f}(t)$$

$A(t)$ is an $n \times n$ matrix
$\vec{X}(t)$ and $\vec{f}(t)$ are $n \times 1$ vectors

If $\vec{f}(t) \equiv \vec{0}$, the system is **homogenous**, i.e. $\vec{X}'(t) = A(t)\vec{X}(t)$

Example:

$$x' = 3x - 2y$$
$$y' = x$$
$$z' = -x + y + 3z$$

$$\Rightarrow \quad \vec{X}' = \underbrace{\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{pmatrix}}_{\boldsymbol{A}} \vec{X} \qquad \vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

It may be easily verified that $\vec{x}_h = \begin{pmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}$ is a solution to the system

$$x' = 3x - 2y$$
$$y' = x$$
$$z' = -x + y + 3z$$

Actually, it can be easily verified that $\begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}$ and $\begin{pmatrix} e^{t} \\ e^{t} \\ 0 \end{pmatrix}$ are also solutions to the same $\vec{X}_h' = A\vec{X}_h$

linear combinations of these will also be solutions

Similarly, for the **non-homogenous ODE**

$$x' = 3x - 2y + 2 - 2e^t$$
$$y' = x - e^t$$
$$z' = -x + y + 3z + e^t - 1$$

⇒

$$\overrightarrow{X}' = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{pmatrix} \overrightarrow{X} + \begin{pmatrix} 2 - 2e^t \\ -e^t \\ e^t - 1 \end{pmatrix}$$

$$\vec{X}_P = \begin{pmatrix} e^t \\ 1 \\ 0 \end{pmatrix}$$

Particular solution of the system

CHECK!

The Superposition Principle for Homogenous Linear DE Systems

If $\vec{x}_1(t), \vec{x}_2(t), \ldots\ldots\ldots \vec{x}_n(t)$ are linearly independent solutions to the homogenous equation $\vec{X}'(t) = A(t)\vec{X}(t)$ then any linear combinations of these, i.e.

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \ldots\ldots\ldots + c_n\vec{x}_n(t)$$

is also a solution to that equation for any set of real constants $c_1, c_2, \ldots\ldots\ldots, c_n$

Using this Superposition Principle and the homogenous and particular solutions obtained earlier -

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t) = c_1\begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix} + c_2\begin{pmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + c_3\begin{pmatrix} e^{t} \\ e^{t} \\ 0 \end{pmatrix} + \begin{pmatrix} e^{t} \\ 1 \\ 0 \end{pmatrix}$$

Linear combination of the three independent solutions of the homogenous equation

particular solution

We need to show that $\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}, \vec{x}_2 = \begin{pmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}, \vec{x}_3 = \begin{pmatrix} e^{t} \\ e^{t} \\ 0 \end{pmatrix}$ are linearly independent on $(-\infty, \infty)$

Step 1: Choose a point, say $t_0 = 0 \in (-\infty, \infty)$

Step 2: Calculate $\vec{x}_1(t_0), \vec{x}_2(t_0), \vec{x}_3(t_0)$ and form the column space matrix

$$C = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

The columns of ***C*** are obviously independent but we will confirm that in the next slide by computing $rref\ (c)$

Step 3: Test for linear independence of the columns of C by computing

$$rref(C)=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Clearly, the column vectors of C must be linearly independent

Alternatively, this could have been shown by calculating and showing that $\det(C) \neq 0$

In general, for a $n \times n$ linear system, we need *n linearly independent solutions* $\vec{X}_1(t), \vec{X}_2(t), ...\vec{X}_n(t)$ to form a basis for the solution space with the general solution to the homogenous system given by

$$\vec{X}_h = c_1\vec{X}_1(t) + c_2\vec{X}_2(t) + ... + c_n\vec{X}_n(t) \qquad c_1, c_1, ...c_n \in \mathbb{R}$$

Fundamental Matrix:

Note that $\overrightarrow{X_h}$ can also be expressed as follows -

$$\vec{x}_h(t) = c_1\begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix} + c_2\begin{pmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + c_3\begin{pmatrix} e^{t} \\ e^{t} \\ 0 \end{pmatrix} \quad \text{OR} \quad \underbrace{\begin{pmatrix} 0 & 2e^{2t} & e^{t} \\ 0 & e^{2t} & e^{t} \\ e^{3t} & e^{2t} & 0 \end{pmatrix}}_{X(t)} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}_{\vec{c}}$$

$$\begin{pmatrix} | & | & | \\ \vec{X}_1 & \vec{X}_2 & \vec{X}_3 \\ | & | & | \end{pmatrix}$$

$X(t)$ **Fundamental Matrix**

$\vec{c}$

Fundamental Matrix $X(t)$ (*continued*)

(i) $\det(X(t)) \neq 0$

One can also show that

$$X'(t) = AX(t)$$

(ii) The **Fundamental Matrix is NOT unique**

A different set of linearly independent solutions will produce a different $X(t)$ but that $\vec{x}_h = X(t)\vec{c}$ would hold

How do we find $\vec{x}_h$ and $\vec{x}_p$ for a System of Linear ODEs?

Consider the Homogenous Solution $\vec{x}_h$ first, i.e. the solution of $\vec{X}' = A\vec{X}$

If we choose solutions of the form $\vec{x} = e^{\lambda t}\vec{v}$,

then substituting in $X'(t) = AX(t)$

gives $\lambda e^{\lambda t}\vec{v} = Ae^{\lambda t}\vec{v}$

Factoring this, we get $e^{\lambda t}(A - \lambda I)\vec{v} = \vec{0}$

Since $e^{\lambda t}$ can never be zero, we need to find λ and $\vec{v}$ such that $(A - \lambda I)\vec{v} = \vec{0}$

But a scalar λ and a non-zero vector $\vec{v}$ satisfying $(A - \lambda I)\vec{v} = \vec{0}$
are the *eigenvalue* and *eigenvector* of the matrix A

Considering the eigenvalues of A, we will have three main cases –

(i) Distinct Real Eigenvalues

(ii) Repeated Real Eigenvalues

(iii) Complex Eigenvalues

for the eigenvalues of A in $X'(t) = AX(t)$

Case (i): $X'(t) = AX(t)$ has real eigenvalues $\lambda_1, \lambda_2, \ldots\ldots \lambda_n$ $\lambda_i \neq \lambda_j$ for $i \neq j$
and the corresponding eigenvectors are $\vec{v}_1, \vec{v}_2, \ldots\ldots, \vec{v}_n$

Note that the eigenvalues are not repeated and, therefore, n independent eigenvectors can be found

For this case, the **General Homogenous Solution** is of the form –

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \ldots\ldots\ldots\ldots + c_n e^{\lambda_n t}\vec{v}_n$$

Note that in the case of repeated eigen values, i.e. $\lambda_i = \lambda_j \;\; i \neq j,$ we will need either independent eigenvectors or generalized eigenvectors, as discussed later

Example Consider the following system of ODEs with initial conditions $x(0) = 3,\ y(0) = 1$

$$\frac{dx}{dt} = -2x + y \qquad \frac{dy}{dt} = x - 2y \quad \Longrightarrow \quad \vec{X}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{X}; \quad \vec{X}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

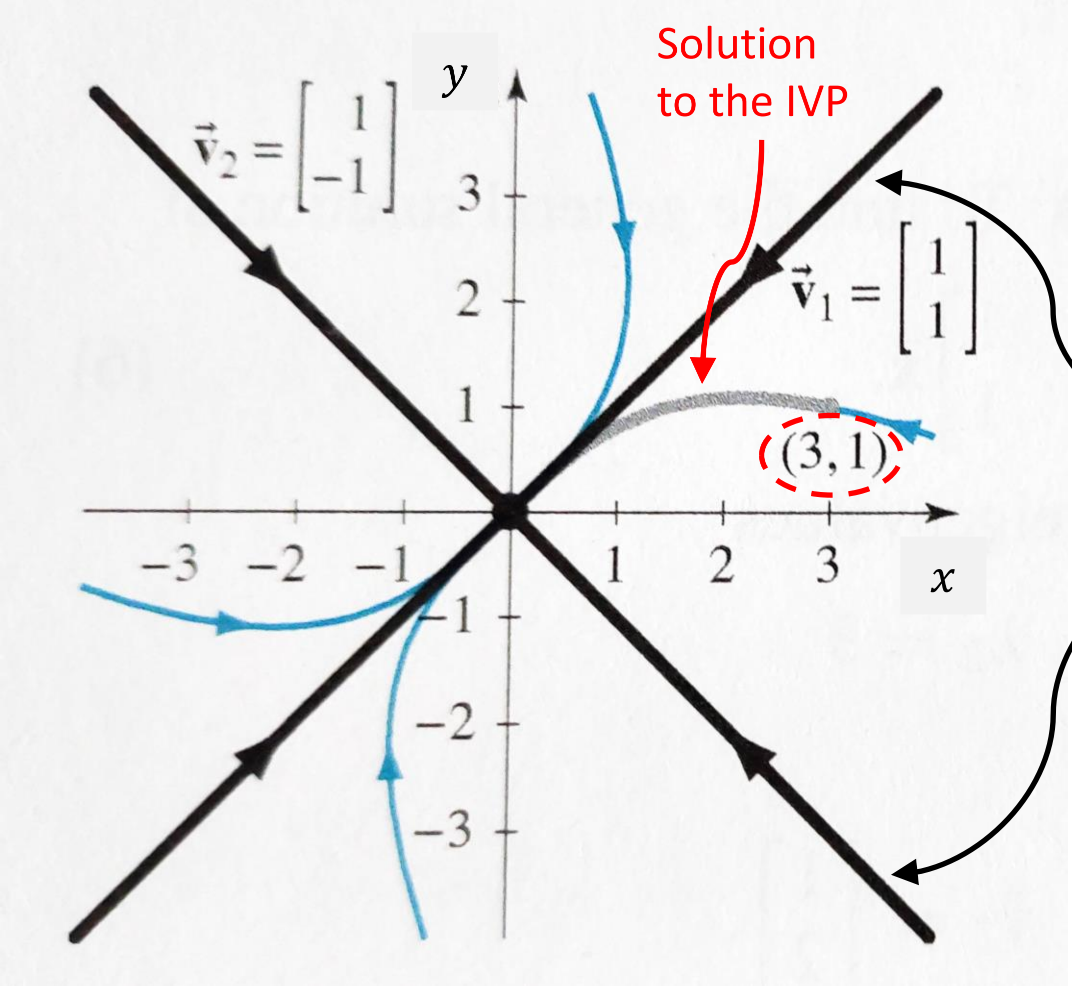
For this, eigenvalues are $\lambda_1 = -1, \lambda_2 = -3$ and eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

General Solution: $\vec{x}(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Using the given initial condition $\vec{X}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow \ c_1 = 2, c_2 = 1$

$$\boxed{\vec{x}(t) = 2e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

Alternatively, $\vec{x}(t) = X(t)\vec{c} = \begin{pmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{-t} + e^{-3t} \\ 2e^{-t} - e^{-3t} \end{pmatrix}$



Phase Portrait
(Stable Equilibrium at origin, solution from (3,1) in grey)

- Trajectories move towards or away from the equilibrium according to the sign of the eigenvalues (-ive or +ive) associated with the eigenvectors
- Along each eigenvector is a unique trajectory called a **SEPRATRIX** that separates the trajectories curving one way from those curving the other way
- The equilibrium occurs at the origin and the phase portrait is symmetric about this point

Case (ii): $X'(t) = AX(t)$ with repeated eigenvalues $\lambda_1, \lambda_2 = \lambda$
and with only one eigenvector $\vec{v}$

Consider only 2×2 case for simplicity

Construct an **additional linear independent vector $\vec{u}$** as follows

Step (i): Find $\vec{v}$ corresponding to λ

Step (ii) Find a new $\vec{u} \neq \vec{0}$ such that $(A - \lambda I)\vec{u} = \vec{v}$

Step (iii) With these, try $\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(t\vec{v} + \vec{u})$

$\vec{u}$ is referred to as the **Generalized Eigenvector** of A

But it is not really an eigenvector as $A\vec{u} \neq \hat{\lambda}\,\vec{u}$

Why this approach works?

Let $\vec{X}_2(t) = e^{\lambda t}(t\vec{v} + \vec{u})$ where we are given that

(a) eigenvalue λ and eigenvector $\vec{v}$ satisfy $(A - \lambda I)\vec{v} = \vec{0}$

and (b) $\vec{X}_1(t) = e^{\lambda t}\vec{v}$ is a solution of $\vec{X}' = A\vec{X}$, i.e. $\vec{X}_1' = A\vec{X}_1$

Show that $\vec{X}_2' = A\vec{X}_2$ if we can find $\vec{u}$ such that $(A - \lambda I)\vec{u} = \vec{v}$

Substituting, $e^{\lambda t}(\vec{v} + \lambda t I\vec{v} + \lambda I\vec{u}) = e^{\lambda t}(tA\vec{v} + A\vec{u})$ and equating the coefficients of $te^{\lambda t}$ and $e^{\lambda t}$ on the LHS and RHS of this equation, we get –

1. Coefficient of $te^{\lambda t}$: $(A - \lambda I)\,\vec{v} = \vec{0}$ This is the original eigenvalue equation that we already had

2. Coefficient of $e^{\lambda t}$: $(A - \lambda I)\vec{u} = \vec{v}$ We need to solve this to find $\vec{u}$ and use it to find $\vec{X}_2(t) = e^{\lambda t}(t\vec{v} + \vec{u})$

Example: Consider $\vec{X}' = A\vec{X} = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} \vec{X}$

Eigenvalue $\lambda = 4$ (repeated)

Eigenvector $\vec{v}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

One solution $\vec{x}_1(t) = e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

If we follow the earlier approach of Lecture 1 of Module 3 then we should try our second solution as $\vec{x}_2(t) = te^{4t}\vec{v}$. However, substituting this $\vec{x}_2(t)$ in $\vec{X}' = A\vec{X}$, we find that this does not work!

See Example 6, pg. 363 of Farlow textbook

Example: Consider $\vec{X}' = A\vec{X} = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} \vec{X}$

Eigenvalue $\lambda = 4$ (repeated)

Eigenvector $\vec{v}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

One solution $\vec{x}_1(t) = e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Instead, we try a Generalized Eigenvector $\vec{u}$ such that $\vec{x}_2(t) = e^{4t}(t\vec{v} + \vec{u})$ is a solution to $\vec{x}_2' = A\vec{x}_2$

This can be simplified to (1) $(A - 4I)\vec{v} = \vec{0}$ and (2) $(A - 4I)\vec{u} = \vec{v}$ by equating the coefficients of e^{4t} and te^{4t} on both sides of $\vec{x}_2' = A\vec{x}_2$

Here (1) is the original eigenvalue equation for $\lambda = 4$ and $\vec{v}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and will not give us anything new

For (2), $(A - 4I)\vec{u} = \vec{v} \Rightarrow \begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Rightarrow 2u_1 + u_2 = -1$

Choosing $u_1 = K\ (say) \Rightarrow u_2 = -2K\text{-}1$ or $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = K\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

Therefore, $\vec{x}_2(t) = te^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + Ke^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{4t}\begin{pmatrix} 0 \\ -1 \end{pmatrix} \Rightarrow$ $\vec{x}_2(t) = te^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{4t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

We drop the middle term as that is just a multiple of our first solution

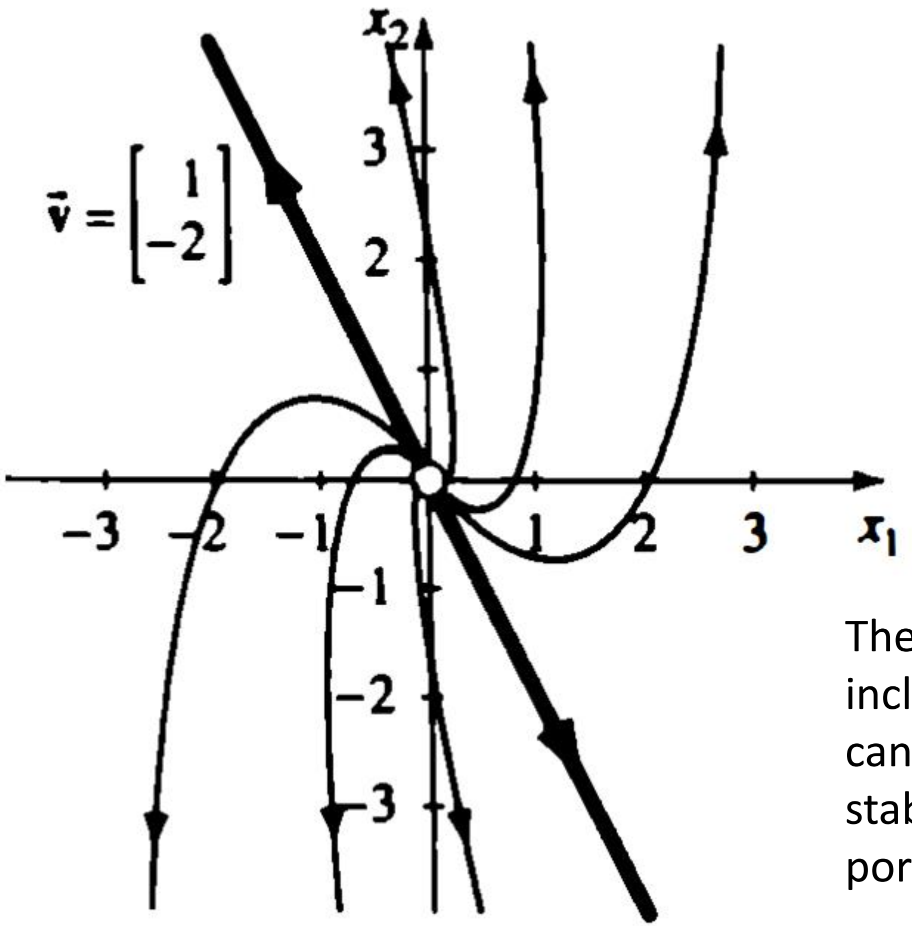
The two solutions are then -

$$\vec{x}_1(t) = e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

and $$\vec{x}_2(t) = e^{4t}\begin{pmatrix} t \\ -2t-1 \end{pmatrix}$$

Final Solution

$$\vec{x}(t) = c_1 e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} t \\ -2t-1 \end{pmatrix}$$



The generalized eigenvector $\vec{\boldsymbol{u}}$ includes a variable t and so cannot be drawn as a second stable vector on the phase portrait

Phase Portrait with
(a) Unstable Equilibrium at the origin
(b) Double Eigenvalue at $\lambda_1 = \lambda_2 = 4$
(c) A single eigenvector

Subsequent Lectures:

(i) Complex Eigenvalues

(ii) Particular solutions $\vec{X}_p$ for systems of linear ODEs

(iii) Phase Portraits and Stability Analysis