There are a total of 6 pages in this project handout.

Project 2
Module 3

Tuning a radio: from Tansen to Baba Sehgal

Prologue

It is the 1990s Chandigarh. Mohua Baweja is a sophomore year student at the Electrical Engineering department of Vriksha University. She is the protagonist of our play. This semester she is taking a class on circuits and one on differential equations. She is also fond of different genres of music, especially rock. Recently, she has learnt in her circuits class to make a radio tuner using a simple RLC electrical circuit. Tonight she is planning to build a simple radio tuner so that she can tap into a live concert by Baba Sehgal and his company who are playing at the rose garden. The radio station at Chandigarh will also be relaying Tansen at the same time at a different frequency.

However, there is a twist in the story. Bhairon Chand Shekhawat is an Applied Mathematics graduate student and lives next door to Mohua Baweja. He is the villain of our play. He is not very keen on rock and instead plans to study for his next month preliminary exam. Loud music from next door will likely spoil his study plans, so he has a back up. He plans a sabotage and quietly replaces the *linear* resistor which Mohua intends to use with an identically looking *non linear* transistor. He is banking on the transistor to skew the stability of the RLC circuit, thereby keeping a check on Mohua and her rock-n-roll party tonight.

3.3.1 Introduction

Goals of the lab

This lab should help you achieve the following goals:

- basic understanding of modeling using differential equations,
- write a higher order differential equation as a system of first order differential equations and then represent the same in matrix-vector form,
- analyze the stability of the system by eigenvalue analysis,
- write the solution as a linear combination of eigenvectors and then analyze long term behavior of the system,
- derive information about the system behavior from time domain plots and phase portraits,
- compare the phase portraits between homogeneous and non-homogeneous cases,
- analyze the effect of non linearity on the stability of the system.

Software

It is advisable, for your own benefit, that you write your own matlab script files and functions for each of the tasks and rely as less as possible on code available on the web. Engineering is as much about intuition as problem solving skills. While problem solving skills can be enhanced by deeper mathematical understanding, intuition may be built by playing with your code and exploring the parameter space of the system; if you are lucky such exploration may lead to interesting and novel observations which should then arouse new mathematical questions. So, write your own code and learn to intelligently play with it!

Report

Your report should present your thought process, observations and conclusions in a very clear and professional manner. It is always a good idea to turn in your work as a pdf file with all figures labelled, equations numbered (equations should be written using a good math editor) and the report categorized into sections and subsections.

3.3.2 The case of Mohua Baweja's radio tuner

Modeling using a series RLC circuit

Mohua plans to use a simple series RLC circuit to tune into the carrier frequency transmitted from the live concert at the rose garden. A schematic of such a circuit is shown in Figure 3.1. The nomenclature of a series RLC circuit is based on the fact that the individual electrical components (resistor, inductor and a capacitor) are placed in series. The voltage $v_i(t)$ is the instantaneous source signal or driving signal that drives the circuit. (**Note:** you may think of $v_i(t)$ as the *known* voltage signal from the radio antenna.)

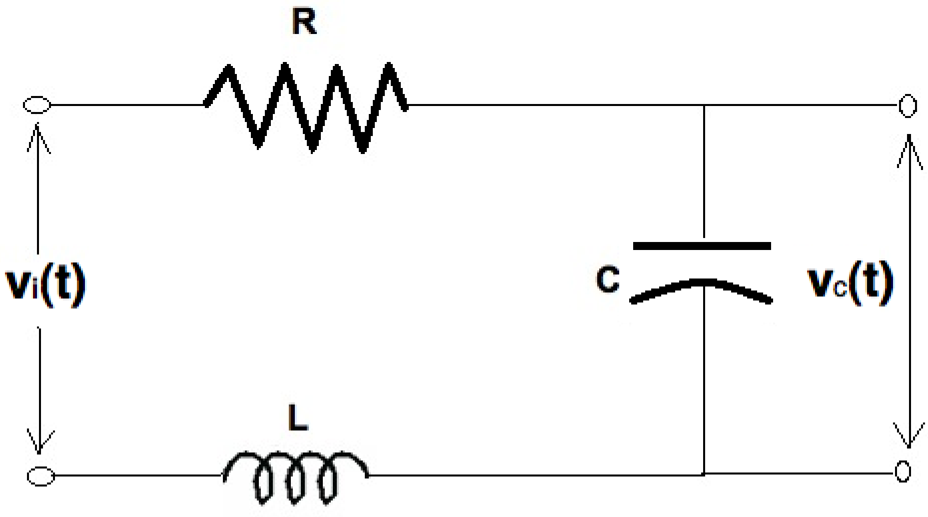

Figure 3.1: Radio tuner: a simple RLC series circuit

The RLC circuit has a natural frequency that depends on the parameters R, L and C. When the frequency of the circuit matches with the frequency of the carrier signal, resonance is attained and one can tap into the frequency of the source signal. Mohua intends to use this simple concept to tap into her desired frequency from the radio channel at the rose garden. To keep her calculation simple, she plans to use a resistor with $R = 0.1\Omega$ and an inductor with $L = 1H$. (*Ohm* or Ω is the unit of electrical resistance and *Henry* or H is the unit of inductance). This way she only has to worry about the capacitance which is now her sole tuning parameter.

She studies her circuits notes and observes that the source voltage, $v_i(t)$ should be equal to the cumulative voltage drops across each of the resistor, capacitor and the inductor as per the celebrated *Kirchoff's law* as follows:

$$v_i(t) = v_R + v_C + v_L \tag{3.1}$$

Moreover, she knows that the voltage drops across each of these electrical elements is given by the following:

$$v_R = Ri(t), \quad v_C = \frac{q(t)}{C}, \quad v_L = L\frac{di(t)}{dt} \tag{3.2}$$

where $q(t)$ is the instantaneous charge in the capacitor and $i(t) = \frac{dq}{dt}$ is the instantaneous current flowing through the circuit.

Questions:

1. Using equations (3.1) and (3.2), obtain a linear differential equation model of the circuit that Mohua plans to use. Note that this differential equation may be expressed in terms of $q(t)$.

2. Suppose $v_i(t)$ is a known differentiable function of t, obtain a second order linear differential equation in terms of $i(t)$.

3. Now express the *homogeneous* part of this higher order linear ODE as a system of first order ODEs and then represent the same in *matrix-vector* notation.

3.3.3 Stability of the circuit

Intuitively, for the circuit to be stable Mohua expects that at any given time the value of $i(t)$ should not blow up or become unbounded. She plans to check this in two ways:
(1) analytically, by doing an eigenvalue analysis. For convenience, Mohua sets $v_i(t) = 0$ during her analysis;
(2) numerically, by using the matlab *ode45* solver to crank her ODE with initial conditions, $i(0) = 0, i'(0) = 1$ and plotting $i(t)$ and $i'(t)$ as time elapses. Again, Mohua sets $v_i(t) = 0$.

Questions:

1. Perform both exercise (1) and (2) as described above and comment on whether the conclusions from each match up.

2. Why do you think setting $v_i(t) = 0$ might have been a reasonable thing to do for the exercise?

3. Justify Mohua's choice of the initial conditions in exercise (2).

4. Provide a phase portrait plot and comment on your observation.

3.3.4 Tapping into Baba Sehgal's channel

Now, as mentioned before that there were two different frequencies being transmitted on that fateful evening, viz. Baba Sehgal's at frequency 1 (rad/sec) and Tansen at frequency 5 (rad/sec) (don't worry about the awkwardness of the frequency values; if this bothers you assume units in Hz and think of time elapsing slower than usual). This means that the driving voltage signal, $v_i(t) = -\cos t - \frac{4}{5}\cos 5t$.

So how does Mohua pick up the rock-n-roll channel while still keeping the signal from the Tansen station reasonably suppressed? She looks at her circuits notes again and observes an expression for current through a series RLC circuit:

$$I = \frac{V}{Z}\cos(\omega t + \phi) \tag{3.3}$$

where $V = |v_i(t)|$, $Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}$ is the *impedance* of the circuit, ω is the frequency of the driving signal in rad/sec and ϕ is the phase lag between the voltage and the current phasors.
Mohua figures out that to derive maximum power from a particular channel she must maximize the current flowing through the circuit when the frequency ω is the one from her desired radio channel.

Questions:

1. From the fact provided in the preceding paragraph, what value of the tuning parameter, C must Mohua choose to tap into Baba Sehgal's concert?

2. Use the value of C obtained above and provide an expression for $i(t)$ that solves the governing equation obtained in section 2.1.1 (2) and analyze the long term behavior of this solution by commenting on the dominant terms as $t \to \infty$. (*hint:-* you do not have to find the scaling constants that appear in the solution of the homogeneous part of the ODE, and for the particular solution you may unleash the *method of undetermined coefficients*.)

3. Justify, by analyzing the solution for $i(t)$ obtained above, that this way Mohua was indeed able to tap into the rock-n-roll frequency by reasonably suppressing the frequency from the Tansen channel.

4. Support your analytical claim by a time domain plot of the circuit response that clearly shows that the long run behavior is exclusively influenced by the rock-n-roll signal with your choice of C.

5. What might be a good value for C if Mohua instead chose to tune into Tansen?

3.3.5 The case of Bhairon Chand Shekhawat's non-linear transistor: sabotage!

The non linear differential equation model of the radio tuner

We now turn to what actually happened on that fateful evening of 1997, the resistor has been replaced by Bhairon with a transistor which has non linear characteristics. We begin by looking at the behavior of the circuit in the absence of any driving signal (i.e. $v_i(t) = 0$). We saw earlier (provided you did your math right) that if it was indeed the resistor which Mohua used, the circuit was found to be stable. Depending upon how you chose the state variables, the system of linear first order ODE was:

$$i' = \frac{1}{L}(-Ri + v) \tag{3.4}$$

$$v' = -\frac{1}{C}i \tag{3.5}$$

where $v = v_C$ is the voltage across the capacitor. (**Note:** this does not have to tally with the system of equations you might have used earlier, this is just one possibility).

However, the resistor has been replaced by the transistor and hence the equation is slightly different, and is given below.

$$\frac{di}{dt} = i' = \frac{1}{L}(-F(i) + v) \tag{3.6}$$

$$\frac{dv}{dt} = v' = -\frac{1}{C}i \tag{3.7}$$

which can be rescaled and written in a more convenient form as follows:

$$\frac{dI}{d\tau} = I' = V - \mu f(I) \tag{3.8}$$

$$\frac{dV}{d\tau} = V' = -I \tag{3.9}$$

where, $\mu = \sqrt{C/L}$, $I = \frac{i}{i_{ref}}, V = \frac{v\sqrt{\frac{C}{L}}}{i_{ref}}, \tau = \frac{t}{\sqrt{LC}}$is the new time scale, i_{ref} = chosen reference current and $f(i) = F(i_{ref}i) = \frac{i^3}{3} - i$ is the famous *Van der Pol* function reminiscent of a transistor. Clearly, $f(\cdot)$ introduces non linearity in the circuit. Does this retain the stable characteristics of the erstwhile series RLC circuit which Mohua had intended to take advantage of?
Note:- for all practical purposes use equations (3.8) and (3.9) for your analysis and answering the questions below.

Questions:

1. Find the equilibrium point(s) of the non linear model (3.8) - (3.9).

3.3.6 A few words on non-linear stability analysis

Recall that for a linear system, one which can be expressed in the form $\vec{x}' = A\vec{x}$ where A is the system matrix; we can perform an eigenvalue analysis of the system matrix, A and thereby analyze the stability of the system. The issue with nonlinear systems is that they cannot be expressed in the linear form $\vec{x}' = A\vec{x}$ precisely because of the presence of nonlinear terms.

One way around this problem is to *linearize* the system about the equilibrium point(s) and then exploit linear theory to comment on the stability of the system near the equilibrium points. Now let us present this argument formally! Consider the following non linear system:

$$x' = h(x, y), \quad y' = g(x, y) \tag{3.10}$$

Let (x_e, y_e) be the equilibrium points obtained by setting the respective derivative terms equal to zero. Since, we have agreed to restrict our analysis close to the equilibrium points, we make the following transformation,

$$u = (x - x_e) \quad \text{and} \quad v = (y - y_e) \tag{3.11}$$

which implies

$$u' = x' \quad \text{and } v' = y'$$

because (x_e, y_e) is constant for the given system. Now, to obtain the *linearized* system, we *Taylor* expand $h(x, y)$ and $g(x, y)$ about (x_e, y_e) as follows:

$$h(x, y) = h(x_e, y_e) + (x - x_e)\frac{\partial h}{\partial x}(x_e, y_e) + (y - y_e)\frac{\partial h}{\partial y}(x_e, y_e) + \text{H.O.T.} \tag{3.12}$$

$$g(x, y) = g(x_e, y_e) + (x - x_e)\frac{\partial g}{\partial x}(x_e, y_e) + (y - y_e)\frac{\partial g}{\partial y}(x_e, y_e) + \text{H.O.T.} \tag{3.13}$$

(**Note:** here H.O.T. means 'higher order terms')
Now using the fact that $h(x_e, y_e) = g(x_e, y_e) = 0$ (why?), equation (3.11) and ignoring H.O.T. we obtain the following:

$$u' = u\frac{\partial h}{\partial x}(x_e, y_e) + v\frac{\partial h}{\partial y}(x_e, y_e) \tag{3.14}$$

$$v' = u\frac{\partial g}{\partial x}(x_e, y_e) + v\frac{\partial g}{\partial y}(x_e, y_e) \tag{3.15}$$

which can then be expressed in matrix form:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \mathcal{J}(x_e, y_e) \begin{pmatrix} u \\ v \end{pmatrix} \tag{3.16}$$

where

$$\mathcal{J}(x_e, y_e) = \begin{pmatrix} \frac{\partial h}{\partial x}(x_e, y_e) & \frac{\partial h}{\partial y}(x_e, y_e) \\ \frac{\partial g}{\partial x}(x_e, y_e) & \frac{\partial g}{\partial y}(x_e, y_e) \end{pmatrix}$$

is the *Jacobian* matrix.

Equation (3.16) is now in linear form and we can use our knowledge from linear theory (based on eigenvalue analysis) to analyze the stability of the system near the equilibrium points (x_e, y_e).

Questions:

The non linear model (3.8) and (3.9) is analogous to the system (3.10) if you consider $h(\cdot, \cdot) \equiv V - \mu f(I)$ and $g(\cdot, \cdot) \equiv -I$ and $(x, y) \equiv (I, V)$.

1. Perform a stability analysis of the non linear system (3.8)-(3.9) by linearizing near the equilibrium points and then comment on the nature of stability of these point(s).
2. Suggest a **potential drawback** of performing such a stability analysis of a non linear system by linearizing the model near equilibrium points.
3. Provide a phase portrait plot and comment on what you observe.
4. Some interesting values of μ are $\mu = 1$, $\mu = 20$. Provide a phase plane plot for each case. What change(s) do you observe in the phase plane with these values of μ? Can you mention a consequence of this change? (*hint:* it might be helpful to overlay the graph of the Van der Pol function on your phase plane plot)
5. Based on your analysis and observation above, explain how Bhairon Chand Shekhawat was able to spoil the party for Mohua Baweja?