Lecture (17) 17/4/2019.

Principal Value Integrals & their Applications

Consider the following integral

$$I = \int_{-\infty}^{\infty} f(x)\,dx \quad ; \; f(x) \text{ is a real valued } f^n. \qquad \text{— (17.1)}$$

We say that I converges (in eq (17.1)) if the following 2 ~~integrals~~ limits exist:

$$\lim_{L\to\infty} \int_{-L}^{\alpha} f(x)\,dx \quad \text{and} \quad \lim_{R\to\infty} \int_{\alpha}^{R} f(x)\,dx \quad ; \alpha < \infty.$$

b/c $$I = \int_{-\infty}^{\infty} f(x)\,dx = \lim_{L\to\infty} \int_{-L}^{\alpha} f(x)\,dx + \lim_{R\to\infty} \int_{\alpha}^{R} f(x)\,dx \qquad \text{— (17.2)}$$

§(17.1) Cauchy Principal value at ∞

Version(1) $$I_p := \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx \qquad \text{— (17.3)}$$

It is possible for the integral in eq (17.3) to exist even though the integral in eq (17.1) may not exist.

eg. $f(x) = x$

$$I_p = \lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \frac{x^2}{2}\Bigg|_{-R}^{R} = \lim_{R\to\infty} \frac{R^2}{2} - \frac{R^2}{2} = 0$$

but I is not defined.

Version(2)

$$P\int_a^b f(x)\,dx \equiv ⨍_a^b f(x)\,dx = \lim_{\epsilon \to 0^+} \left[\int_a^{x_0-\epsilon} + \int_{x_0+\epsilon}^b\right] f(x)\,dx \quad (17.4)$$

is useful for calculating $\int_a^b f(x)dx$ when $f(x)$ is singular at $x = x_0$.

We will see a concrete application of the Principal value integral after we state the following 2 results.

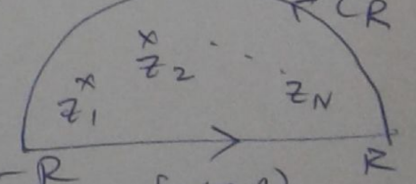
(17.2) **Th^m**:- Let $f(z) = \frac{N(z)}{D(z)}$ be a rational f^n s.t. $\mathrm{Deg}(D(z)) - \mathrm{Deg}(N(z)) \geq 2$;

then $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$ w/ C_R defined as below. #

this above theorem is useful to calculate

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_{C_R} f(z)\,dz = \oint_C f(z)\,dz$$

Often we have to calculate this integral.



fig(17.2)

$z_1, z_2, \ldots z_N$ are the poles (singularities) of $f(z)$

We will then use the following facts

(I) Cauchy Residue th^m
(II) $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$

to obtain

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{Res}\left(f(z); z_j\right)} \quad \text{—(17.5)}$$

$$= 2\pi i \operatorname{Res}\left(f(z); \infty\right) \text{ if } \{z_i\}$$

are isolated s.p.s

§(17.3) (Jordan's Lemma)

If on C_R (see fig (17.2)) we have $f(z) \to 0$ uniformly (or $|f(z)| \to 0$) as $R \to \infty$;

then $\lim_{R\to\infty} \int_{C_R} e^{ikz} f(z)\,dz = 0$; $k > 0$.

We are now ready to study a very important #

§(17.4) result in complex analysis which will illustrate an application of Cauchy Principal value integral & Th^m(17.2)

Th^m(17.4)

(Sokhotski - Plemelj Formula)

more specifically $f(x)$ belongs to Schwarz f^n sp.

Let $f(x)$ satisfy the Holder's condition $|f(x) - f(y)| \le C\|x-y\|^{\alpha}$; $\alpha, C > 0$

then

$$\boxed{\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{f(x)\,dx}{x \pm i\epsilon} = \fint_{-\infty}^{\infty} \frac{f(x)}{x}\,dx \mp i\pi f(0)} \quad \text{—(17.6)}$$

Note $f(0) = \int_{-\infty}^{\infty} \delta(x) f(x)\,dx \equiv \int_{-\infty}^{\infty} f(x)\delta(x)\,dx$

this result is credited to Julian Sochocki (Russian - Polish mathematician) and Josip Plemelj (Slovenian)

and is a main ingredient of the solutions to Riemann Hilbert problems.

The proof of this theorem runs a few pages long; so we will prove only a very special case when $f(x) = 1$.

Proof: When $f(x) = 1$

We will prove:-

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{dx}{x+i\epsilon} = ⨍_{-\infty}^{\infty} \frac{dx}{x} - i\pi \int_{-\infty}^{\infty} \delta(x)\, dx \quad \text{—— (i)}$$

Often, the result is written in abbreviated form as follows

$$\lim_{\epsilon \to 0} \frac{1}{x+i\epsilon} = P\left(\frac{1}{x}\right) - i\pi\, \delta(x).$$

$$\text{L.H.S.} = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{x-i\epsilon}{(x+i\epsilon)} \frac{1}{(x-i\epsilon)} dx$$

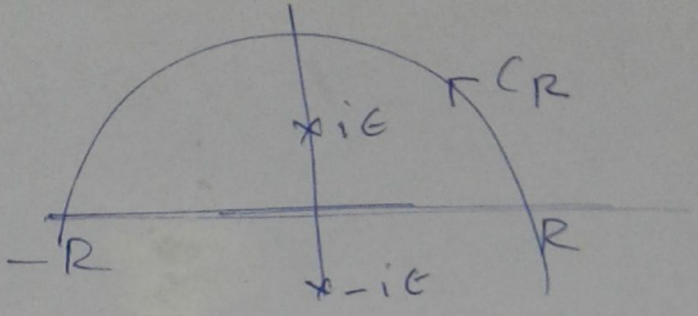
$$= \lim_{\epsilon \to 0} \lim_{\delta \to 0} \left(\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} \frac{x}{x^2+\epsilon^2} dx + \int_{-\delta}^{\delta} \frac{x}{x^2+\epsilon^2} dx \right)$$

$$- i \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{\epsilon}{x^2+\epsilon^2} dx$$

$$= I_1 + I_2 + I_3 \quad \text{—— (ii)}$$

where $I_2 = 0$ b/c the integrand $\frac{x}{x^2+\epsilon^2}$ is an odd f^n of x.

I_3 is calculated by using Th^m (17.2) & Cauchy Residue Th^m.



$$\oint_C \frac{\epsilon}{z^2+\epsilon^2}\,dx = 2\pi i \,\mathrm{Res}\left(\frac{\epsilon}{z^2+\epsilon^2};\ z_0 = i\epsilon\right)$$

$$= 2\pi i\,\mathrm{Res}\left\{\frac{1}{2i}\left(\frac{1}{z-i\epsilon} - \frac{1}{z+i\epsilon}\right);\ i\epsilon\right\}$$

$$= 2\pi i \times \frac{1}{2i} = \pi$$

$$\mathbin{/\!/}\ \lim_{R\to\infty}\left\{\int_{-R}^{R}\frac{\epsilon}{x^2+\epsilon^2}\,dx + \int_{C_R}\frac{\epsilon}{z^2+\epsilon^2}\,dz\right\}$$

$$\text{i.e. } \lim_{R\to\infty}\int_{-R}^{R}\frac{\epsilon}{x^2+\epsilon^2}\,dx + \lim_{R\to\infty}\cancel{\int_{C_R}\frac{\epsilon}{z^2+\epsilon^2}\,dz} = \pi$$

0 b/c $\mathrm{Deg}(z^2+\epsilon^2) = 2$
$\mathrm{Deg}(\epsilon) = 0$
so th^m (17.2) applies

$$\text{thus } \oint_C \frac{\epsilon}{z^2+\epsilon^2}\,dz = \int_{-\infty}^{\infty}\frac{\epsilon}{x^2+\epsilon^2}\,dx = \pi \quad \text{——— (iii)}$$

$$\Rightarrow I_3 = -i\pi = -i\pi\int_{-\infty}^{\infty}\delta(x)\,dx$$

Now we will calculate I_1.

We choose δ s.t. it vanishes to 0 at the same rate as $\epsilon \to 0$; i.e. set $\delta = \epsilon$. This is possible b/c the use of δ is completely artificial & a matter of construction.

$$I_1 = \lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \frac{x^2}{x^2+\epsilon^2}\,\frac{dx}{x} \qquad \text{b/c } \delta = \epsilon \text{ is set.}$$

$$\equiv \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{x^2}{x^2+\epsilon^2}\,\frac{dx}{x}$$

this is a "cool" representation (notation) & you must learn it

$$= \underbrace{\lim_{\epsilon \to 0} \int_{|x|>\epsilon} \left(\frac{x}{x^2+\epsilon^2} - \frac{1}{x} \right) dx}_{\text{Call this } I_{>\epsilon} \text{ (say)}} + \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{1}{x}\,dx \quad \text{— (iv)}$$

$$I_{>\epsilon} = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{-\epsilon^2}{(x^2+\epsilon^2)\,x}\,dx \;\xrightarrow[x=\epsilon u]{\text{substitute}}\; \lim_{\epsilon \to 0} \int_{|u|>1} \frac{-\epsilon^2\,\epsilon\,du}{\epsilon^2(u^2+1)\,\epsilon u}$$

$$= -\lim_{\epsilon \to 0} \int_{|u|>1} \frac{du}{u(u^2+1)}$$

$= 0$ b/c the integrand is an odd f^n of u.

thus
$$I_1 = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{dx}{x}$$

$$= \lim_{\epsilon \to 0} \int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \frac{dx}{x}$$

$$= ⨍_{-\infty}^{\infty} \frac{dx}{x} \quad \text{— (v)}$$

Using eqs. (iii) & (v) in eq (ii); we have.

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{dx}{x+i\epsilon} = ⨍_{-\infty}^{\infty} \frac{dx}{x} - i\pi \int_{-\infty}^{\infty} \delta(x)\,dx \qquad \#$$