System of ODEs.

Q1)
$$\left.\begin{aligned}\frac{dx}{dt} &= -2x + y \\ \frac{dy}{dt} &= x - 2y\end{aligned}\right\}\text{Coupled ODEs}$$

IC: $x(0) = 3$
$y(0) = 1$

$$\vec{X} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\vec{X}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{X}$$

w/ IC: $\vec{X}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Form: $\dfrac{d\vec{x}}{dt} = A\vec{x}$

Here

$$A = \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}$$

eVs EVs (lin. indep.)

distinct real eVs

$$\lambda_1 = -1 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -3 \qquad \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

General soln:

$$\vec{x}(t) = c_1 e^{+\lambda_1 t}\vec{v}_1 + c_2 e^{+\lambda_2 t}\vec{v}_2$$

Plug in IC.

$$\vec{X}(0) = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix}$$

$$\Rightarrow c_1 = 2$$
$$c_2 = 1$$

Next,

Write general soln. in the form

$$\vec{X}(t) = \underbrace{\mathbb{X}(t)}_{\text{Fundamental Matrix}}\vec{C}$$

Fundamental Matrix

(Not unique!)

$$\mathbb{X}(t) = \begin{pmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{pmatrix}$$

$$\vec{C} = \begin{pmatrix}2\\1\end{pmatrix}$$

|Notes| Date

How do we draw all these soln trajectories??

1. Identify eq^m pts. (if any)
2. Draw EVs
   — draw separatrix (arrows — based on sign of evs)
3. Now locate some pts. that you think may be on the soln. trajectories!
   eg. ICs.

$(x,y) = (3,1)$ $\quad \vec{x}' @ (3,1) = \begin{pmatrix} -6+1 \\ 3-2 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$

$x' = -5 < 0$ ← $\Rightarrow$ h-nullcline

$y' = 1 > 0$ ↑ $\Rightarrow$ V-nullcline

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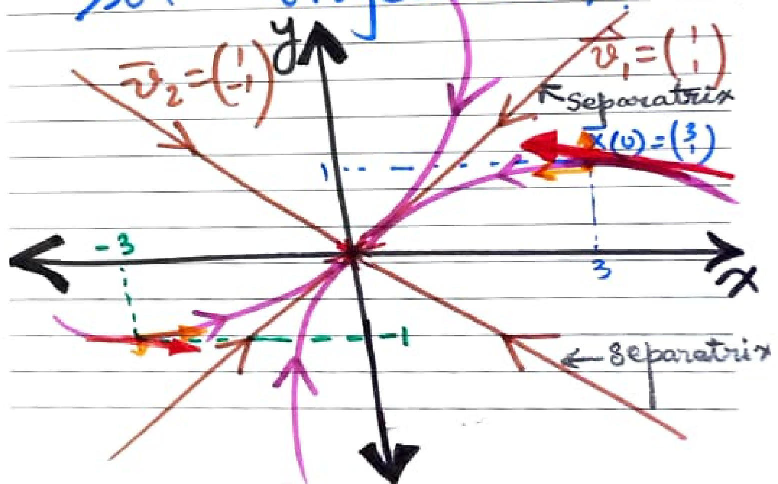
$$\vec{x}(t) = 2\overset{\lambda_1}{e^{-t}}\underset{\vec{v}_1}{\begin{pmatrix}1\\1\end{pmatrix}} + 1\overset{\lambda_2}{e^{-3t}}\underset{\vec{v}_2}{\begin{pmatrix}1\\-1\end{pmatrix}}$$

$\lambda_1, \lambda_2 < 0$
$\Rightarrow$ stable eq^m pt.

$\frac{dx}{dt}, \frac{dy}{dt} = 0$
(1) Where are they?

(2) How do we represent this soln on the phase portrait?
i.e. Can we draw some soln trajectories??



Q2) What if 2 evs are repeated & we have Only 1 EV corresponding to ~~the~~ repeated evs ??

ev: $\lambda_1 = \lambda_2 = \lambda$

EV: $\vec{v}$

Strategy :-

Construct an additional lin. indep. EV $\vec{u}$

generalized EV of A w.r.t. λ

Steps

(i) find $\vec{v}$ corresponding to $\vec{u}$

(ii) find (construct) a new $\vec{u} \neq \vec{0}$ s.t.

$$(A - \lambda I)\vec{u} = \vec{v}$$

(iii) Soln: $\vec{X}(t) = c_1 e^{\lambda t} \vec{v} + c_2 e^{\lambda t} (t\vec{v} + \vec{u})$

Q2)

Solve the system

$$\vec{X}' = A\vec{X} = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

ev: $\lambda = 4$ (repeated)

EV: $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow$ one soln.

$$\vec{X}_1(t) \alpha e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Now, let's find the gen. EV $\vec{u}$

$$(A - \lambda I)\vec{u} = \vec{v}$$

$$\Rightarrow (A - 4I)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$(A-4I)\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}$$

$$\begin{pmatrix}-2 & -1\\4 & 2\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}$$

$-2u_1-u_2=1$ ← same

$4u_1+2u_2=-2 \Rightarrow 2u_1+u_2=-1$

So choose $u_1=k$

$u_2=-2k-1$

$$\therefore \bar{u}=k\begin{pmatrix}1\\-2\end{pmatrix}+\begin{pmatrix}0\\-1\end{pmatrix}$$

So 2^{nd} soln.

ignore this b/c same as $\vec{x}_1(t)$

$$\vec{x}_2(t)=t\bar{v}+\bar{u}=te^{4t}\begin{pmatrix}1\\-2\end{pmatrix}+ke^{4t}\begin{pmatrix}1\\-2\end{pmatrix}+e^{4t}\begin{pmatrix}0\\-1\end{pmatrix}$$

* Only 1 Separatrix $\vec{v}$

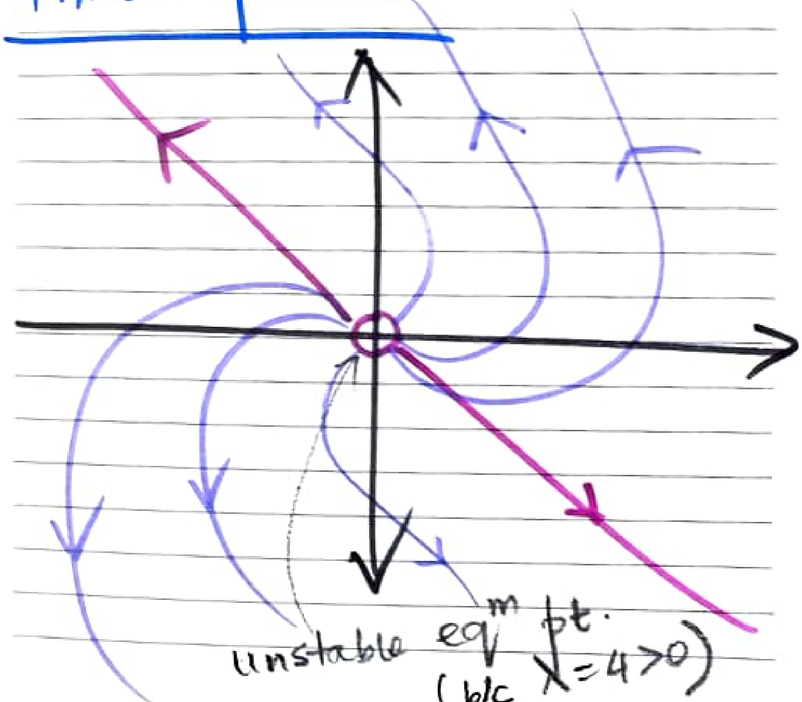
* Why could we not draw the other separatrix involving gen. EV $\vec{u}$?

b/c $\vec{u}$ is a f^n of time (t) (& ph portrait is a static snapshot) & cannot display an evolving separatrix

full soln

$$\vec{x}(t) = c_1 e^{4t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2 e^{4t}\begin{pmatrix}t\\-2t-1\end{pmatrix}$$

Phase portrait



Q) Why bother learning to solve sys. of ODEs in matrix vector form ??

Ans) higher order ODEs w/ const. coeff. can be reduced to a system of 1st order ODEs by simple subn.

$y \longrightarrow x_1$

$y' \longrightarrow x_2$

$y'' \longrightarrow x_3$

and so on ...

Recall eg. from last lecture!

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$$

IC: $y(0)=1$, $y'(0)=0$

sub. $x_1 = y$

$x_2 = \frac{dy}{dt} = x_1'$

$x_3 = \frac{d^2y}{dt^2} = x_2'$

$x_1' = x_2 = y'$

$x_2' = y'' = -5y' - 6y$

$= -5x_2 - 6x_1$

$$\vec{X}' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\vec{X}' = A \vec{X}$$

$$A = \begin{pmatrix} 0 & 1 \\ -6 & -5 \end{pmatrix}$$

eVs: $\lambda_1 = -2$
$\lambda_2 = -3$

recall roots of the characteristic eqn.

EVs: $\vec{v}_1 = \begin{pmatrix} K_1 \\ -2K_1 \end{pmatrix} = K_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\vec{v}_2 = \begin{pmatrix} K_2 \\ -3K_2 \end{pmatrix} = K_2 \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

Soln:-

$$\bar{X}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = K_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + K_2 e^{-3t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

use ICs to find K_1, K_2