Q-1 (f) Gambler's ruin (this is a problem of on law of total probability/ Conditional probability).

Total pts = 5

R := event that the gambler loses all her money

$p_x := P(R \mid \text{currently has } ₹\, x)$; $x = 0, 1, \ldots, N$

Boundary conditions

$$\left.\begin{aligned} p_0 &= 1 \\ p_N &= 0 \end{aligned}\right\} \quad 1 + 1 \text{ pts.}$$

Note that the law of total probability gives

$P(R \mid \text{has } ₹ x) = P(R \mid 1^{st} \text{ toss is H; has } ₹x) \times P(1^{st} \text{ toss is H} \mid \text{has } ₹x) + P(R \mid 1^{st} \text{ toss is T; has } ₹x) \times P(1^{st} \text{ toss is T} \mid \text{has } ₹x)$

$= P(R \mid \text{has } ₹ x+1)\frac{1}{2} + P(R \mid \text{has } ₹ x-1)\frac{1}{2}$

because

$p_x = P(R \mid \text{has } ₹\, x)$

using law of total probability $\stackrel{\curvearrowright}{=} \frac{1}{2} P(R \mid \text{has } ₹\, x+1) + \frac{1}{2} P(R \mid \text{has } ₹\, x-1)$ 1 pt.

$p_x = \frac{1}{2} p_{x+1} + \frac{1}{2} p_{x-1}$ —— 1 pt.

$\Rightarrow \underbrace{2p_x = p_{x+1} + p_{x-1}}_{\text{Recurrence Relation}}$

i.e. $p_{x+1} = 2p_x - p_{x-1}$

$\because p_0 = 1$

$p_2 = 2p_1 - 1$

$p_3 = 2p_2 - p_1 = 2(2p_1 - 1) - p_1 = 3p_1 - 2$

$p_4 = \cdots = 4p_1 - 3$

By inspection, the pattern reveals.

$p_x = xp_1 - (x-1)$ —— (i)

$\therefore p_N = Np_1 - (N-1) \Rightarrow 0 = Np_1 - (N-1)$

$\Rightarrow p_1 = \left(1 - \frac{1}{N}\right)$ —— (ii)

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Substitute eqn. (ii) in eq (i)

$$p_x = xp_1 - (x-1)$$

$$= x\left(1 - \frac{1}{N}\right) - (x-1)$$

$$= x - \frac{x}{N} - x + 1$$

$$\boxed{p_x = 1 - \frac{x}{N}}$$ —— 1 pt.

this is the probability of gambler's ruin.

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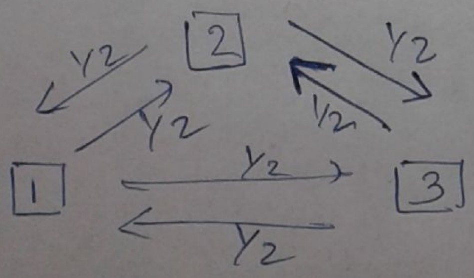
Q. 2 (a)

Chapman - Kolmogorov eqn.

$$p_{ij}^{m+n} = \sum_{k \in S} p_{ik}^{m} p_{kj}^{n}$$; for $m, n \geq 0$ and constant integers.

—— 1 pt.

Q. 2 (c)



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States space = {1, 2, 3} where the vertices are labelled 1, 2, 3,

—— 1 pt.

Probability transition matrix = $\mathbb{P}$ = $\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1/2 & 1/2 \\ 2 & 1/2 & 0 & 1/2 \\ 3 & 1/2 & 1/2 & 0 \end{array}$

—— 1 pt.

We wish to find $\mu_1(2) \equiv \mu_{12}$ i.e. mean no. of steps to get to V2 from V1.

By def^n & results from the theory of hitting times in discrete time Markov chains, we have

$$\mu_{i2} = \begin{cases} 0 & ; i = 2 \\ 1 + \sum_{j \neq 2} p_{ij} \mu_{j2} & ; i \neq 2 \end{cases}$$

1 pt

$$\begin{cases} \mu_{22} = 0 \\ \mu_{12} = 1 + \frac{1}{2}\mu_{22} + \frac{1}{2}\mu_{32} = 1 + \frac{1}{2}\mu_{32} \\ \mu_{32} = 1 + \frac{1}{2}\mu_{22} + \frac{1}{2}\mu_{12} \\ \quad = 1 + \frac{1}{2}\mu_{12} \\ \quad = 1 + \frac{1}{2}\left(1 + \frac{1}{2}\mu_{32}\right) \end{cases}$$

$$\Rightarrow \mu_{32} = 2.$$

1 pt

$$\mu_{12} = 1 + \frac{1}{2}\mu_{32} = 2 \text{ steps.}$$

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