Practical challenges in Gauss Elimination (1)

Consider the following example. We will apply Gauss elimination w/ <u>4-digit arithmetic + rounding</u>

$$R_1: \quad 0.003000x_1 + 59.14x_2 = 59.17$$
$$R_2: \quad 5.291x_1 - 6.130x_2 = 46.78$$

exact sol's. $x_1 = 10.00$, $x_2 = 1.00$

The first pivot coeff 0.003 is small!

In order to eliminate x_1 from R_2, we need to multiply $m_{21} = \dfrac{5.291}{0.003} = 1763.67 \approx 1764.$

$$R_2 : R_2 - m_{21} R_1 \xRightarrow{\text{Reality}} \begin{cases} 0.003000x_1 + 59.14x_2 = 59.17 \\ -104309.5738\, x_2 = -104309.5739 \end{cases}$$

this yields $x_2 \approx 1.000$

Using 4-digit arithmetic

$$0.003000x_1 + 59.14x_2 \approx 59.17$$
$$-104300x_2 \approx -104400.00$$

How?
1764×59.14
$= 104,322.96$
4-digit 104300.00
& etc ....

$$\Rightarrow x_2 = 1.001$$

the error (round-off) has not manifested yet!

Now Backsubstitution:

$$x_1 \approx \frac{59.17 - (59.14)(1.001)}{0.003000} \approx -10.00$$

The small error 0.001 when multiplied by $\frac{59.14}{0.003} = 19713.3$

gives the approx^n to $x_1 = 10.00$

1) So what is the remedy ??

Ans) Do partial pivoting! (Do not work w/ pivotal coeff. that are very very small)

$R_1: \quad 0.003000 x_1 + 59.14 x_2 = 59.17$

$R_2: \quad 5.291 x_1 - 6.130 x_2 = 46.78$

First select

$$\max\{|a_{11}^{(1)}|, |a_{21}^{(1)}|\} = \max\{|0.003000|, |5.291|\}$$

$$= |5.291| = |a_{21}^{(1)}|$$

then swap $R_2 \longleftrightarrow R_1$ to make $a_{21}^{(1)}$ as pivotal coeff. of x_1

$$\begin{cases} 5.291 x_1 - 6.130 x_2 = 46.78 \\ 0.003000 x_1 + 59.14 x_2 = 59.17 \end{cases}$$

The multiplier for this system is

$$m_{21} = \frac{0.003000}{5.291} = 0.00056700$$

Now the reduced system is obtained by

$R_2 : R_2 - m_{21} R_1$

$$5.291\,x_1 - 6.130\,x_2 \approx 46.78$$

$$59.14\,x_2 \approx 59.14$$

gives $x_2 = 1.000$

Correct answers!

and $x_1 \approx \dfrac{46.78 + 6.130 \times 1.000}{5.291} = 10.00$