Lecture (7)

Def^n:-

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Col(A) is a vector space formed by the columns of the matrix and all their linear combinations.

Applications :- w.r.t. $A\vec{x} = \vec{b}$, we have already seen in the previous lecture, why the col(A) & the col^m vectors of the matrix are very important & offer

- algebraic,
- geometric, and
- physical

intuition/meaning/interpretation about the sol^{n}s. of $A\vec{x} = \vec{b}$.

More specifically, we have seen a real life application vis-á-vis the spring-mass system & used the Col(stiffness matrix) as an important framework to glean insight about ~~on~~ the eq^m sol^{n}s. of this system!

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One of the important steps/analysis in that problem was to check if the col^m vectors ~~are~~ form a linearly dependent or lin. independent set. At that time, we relied on inspection & some trial & error strategy. In this lecture, we will formalize a more reliable & convenient approach to deduce the same conclusion. This approach relies on

- elementary row transformations of the matrix
- bringing the matrix to what is known as the "reduced row echelon form" (rref).

We will illustrate both these ideas w/ the help of an example:

Additionally, we will demonstrate an algorithmic strategy to find the soln of $A\vec{x} = \vec{b}$.

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eg ①:- Solve the system of eq^ns:

Augment what? : Ans: A by $\vec{b}$ to form $\tilde{A}$

$$x + 2y + z = 4$$
$$0x + y + 2z = 3$$
$$x + 0y - z = 0$$

OR

$$\underset{A}{\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & -1 \end{pmatrix}} \underset{\vec{x}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{\vec{b}}{\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}}$$

Augmented matrix $\tilde{A}$

System of eq^ns.

(I)

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & -1 & 0 \end{array}\right)$$

Original system of eq^ns:

$$x + 2y + z = 4 \quad ①$$
$$0x + y + 2z = 3 \quad ②$$
$$x + 0y - z = 0 \quad ③$$

(II) $R_3 \to (R_3 - R_1)$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -2 & -4 \end{array}\right)$$

New system of eq^ns:

$eq^n(3)$: $eq(3) - eq(1)$

$$x + 2y + z = 4$$
$$0x + y + 2z = 3$$
$$0x - 2y - 2z = -4$$

Here, I have used eq^n(3) & eq^n(1) to write an alternative eq^n in place of eq^n(3). Doing this does NOT change the problem b/c all the $info^n$ contained in the original 3 eq^ns is retained in the new sys.

Augmented Matrix

$\tilde{A}$

System of eq^{n}s

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(III)

$R_3 \rightarrow R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 2 \end{array}\right]$$

eq(3) : eq(3) + 2eq(2)

$$\begin{cases} x + 2y + z = 4 \\ 0x + y + 2z = 3 \\ 0x + 0y + 2z = 2 \end{cases}$$

Once again the info^n contained in this system is consistent w/ the info^n in the system in Step (II).

(IV) $R_3 \rightarrow \frac{1}{2}R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

eq(3) : $\frac{1}{2}$eq(3)

$$\begin{cases} x + 2y + z = 4 \\ 0x + y + 2z = 3 \\ 0x + 0y + z = 1 \end{cases}$$

Again nothing changes in terms of the info^n / soln being sought.

Augmented matrix $\tilde{A}$

System of $Eq^n s$.

(V) $R_2 \to R_2 - 2R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

eq(2): eq(2) − 2 eq(3)

$$\begin{cases} x + 2y + z = 4 \\ 0x + y + 0z = 1 \\ 0x + 0y + z = 1 \end{cases}$$

By now you should be guessing the intention here: I'm slowly trying to extract the (sol^n) values of $z, y, \ldots$ (in the reverse order of the components of $\vec{x}$).

(VI) $R_1 \to R_1 - 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

eq(1): eq(1) − 2 eq(2)

$$\begin{aligned} x + 0y + z &= 2 \\ 0x + y + 0z &= 1 \\ 0x + 0y + z &= 1 \end{aligned}$$

(VII) $R_1 \to R_1 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

eq(1): eq(1) − eq(3)

$$\begin{aligned} x + 0y + 0z &= 1 \\ 0x + y + 0z &= 1 \\ 0x + 0y + z &= 1 \end{aligned}$$

* Look at the form of the augmented matrix $\tilde{\mathbb{A}}$ in step (VII)

$$\mathbb{A}_T \quad \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \vec{b}_T$$

PIVOTS of $\mathbb{A}$

If we have to read the system of eqs. from this form of $\tilde{\mathbb{A}}$; we would do it as follows.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

i.e. soln:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\mathbb{A}_T \vec{x} = \vec{b}_T$$

* even though $\mathbb{A} \rightarrow \mathbb{A}_T$ and $\vec{b}$ is now b_T i.e. $\vec{b} \rightarrow \vec{b}_T$;

We have shown by writing down the steps earlier that the soln $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ is preserved.

i.e. $\mathbb{A}\vec{x} = \vec{b} \equiv \mathbb{A}_T \vec{x} = \vec{b}_T$

This procedure of finding the soln $\vec{x}$ by deducing the rref of $\mathbb{A}$ is known as the GAUSS-JORDAN ELIMINATION.

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* Infact $A_T = \text{rref}(A)$.

i.e. A_T is the reduced row Echelon form of A.

Cf. def^n in the pdf document / end of this lecture note

and

the elementary row transformations for this problem (not unique steps) are:

$$R_3 \to R_3 - R_1$$
$$R_3 \to R_3 + 2R_2$$
$$R_3 \to \frac{1}{2} R_3$$
$$R_2 \to R_2 - 2R_3$$
$$R_1 \to R_1 - 2R_2$$
$$R_1 \to R_1 - R_3$$

* Once we bring A to $A_T = \text{rref}(A)$, we identify the "pivot" col^m as the lin. independent col^m vectors; infact the pivot col^m vectors are the basis of $\text{Col}(A)$!

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* **Pivot col^{m}s are the basis of col(A)**

the original col^m in A that ~~correspond~~ host the pivot entries of A_T.

in this eg.

$$\begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}, \begin{Bmatrix} 2 \\ 1 \\ 0 \end{Bmatrix}, \begin{Bmatrix} 1 \\ 2 \\ -1 \end{Bmatrix}$$ are the

basis of col(A).

$\dim(\text{col}(A)) = 3$ (in this case).

RANK of a matrix (A) := No. of pivots in RREF(A)

eg (2) Let us consider the sys. of eq. from the spring mass system from the last lecture.

case (III): $A\bar{u} = \bar{f}$

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The augmented matrix

$$\tilde{A} = \left(\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & 1 & 1 \end{array}\right)$$

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$R_2 \to R_2 + R_1$

pivot.

$$\left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right) \Rightarrow 0 = 2 \text{ (Impossibility)}$$

No soln.

$$\text{RREF}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

pivot col^n $\left\{\begin{matrix} 1 \\ -1 \end{matrix}\right\}$

$$\text{Basis of col}(A) = \left\{\begin{matrix} 1 \\ -1 \end{matrix}\right\}$$

$$\dim(\text{col}(A)) = 1$$

<u>Case (ii)</u>

$R_2 \to R_2 + R_1$

$$A_T = \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow 0 = 0 \text{ (fine)}.$$

No unique sol^n.

family of solns.

$$\text{RREF}(A) = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

$$\text{Basis of col}(A) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\dim(\text{col}(A)) = 1$$

Defⁿ

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Reduced Row Echelon Form (RREF)

A matrix is said to be in rref if it satisfies ALL the following conditions :-

1. If a row has non-zero entries, then the first non-zero entry is 1. This is known as the leading 1 or the PIVOT entry of that row.

2. If a colᵐ has a pivot entry, then all the other entries in that colᵐ are 0.

3. If a row contains a pivot entry, then each row above it contains a leading 1 (or pivot) further to the left.

Q) How to convert a matrix to rref/ref?

Ans) Elementary row transformations:
(i) Divide a row by a non-zero scalar.
(ii) Add/Subtract a multiple of a row w/from another
(iii) Swap two rows.

MCQ - quiz ①

Q) So what questions may be asked from today's lecture?

Ans:-

1. Rank of a matrix
2. Find the basis of Col(A)
3. dim(Col(A))
4. How to find soln. to $A\bar{x} = \bar{b}$ using rref(A) i.e. Gauss-Jordan elimination method.

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