25/3/19

Lecture (12): Sequences & Series of Complex functions.

§(12.1) Fundamental ideas of Seq. & Series in $\mathbb{C}$.

We say $f_n(z)$ converges to $f(z)$ on a suitable subset $R \subset \mathbb{C}$ if $\lim_{n \to \infty} f_n(z) = f(z)$.

Just like in the reals, we can establish an $\epsilon - \delta$ definition of above.

Moreover, just like in the reals, we can define an infinite series as an infinite sequence of partial sums.

$$s_n(z) = \sum_{j=1}^{n} b_j(z)$$

$$S(z) = \lim_{n \to \infty} s_n(z) = \sum_{j=1}^{\infty} b_j(z).$$

This is basically a unification (equivalence) of the ideas of series & sequences in mathematics, there is no real distinction between the same.

Uniform convergence :- $s_n(z) \xrightarrow{\text{unif}} s(z)$ if

$$|s_n(z) - s(z)| < \epsilon$$

for some apriori chosen $\epsilon > 0$ & $\forall\, n > N = N(\epsilon)$ & $\forall\, z \in R \subset \mathbb{C}$.

"N does NOT depend on z".

eg. $f_n(z) = \frac{1}{nz}$; $n = 1, 2, \ldots$

$f_n \to 0$ uniformly in $1 \leq |z| \leq 2$ b/c

$$|f_n(z) - f(z)| = \left|\frac{1}{nz} - 0\right| = \frac{1}{n|z|} < \epsilon$$

$$\forall\, n > N(\epsilon) = \frac{1}{\epsilon}.$$

eg $f_n(z) = \frac{1}{nz}$; $n = 1, 2, \ldots$

Converges to 0 (but not uniformly) on $0 < |z| \le 1$

b/c $|f_n - 0| < \epsilon$ only if $n > N(\epsilon, z) = \frac{1}{\epsilon|z|}$ in $0 < |z| \le 1$.

"Uniform convergence" (if there is one) is a very powerful & useful condition

Continuous f^n space on R

Th^m (12.1) :- Let $f_n(z) \in \mathcal{C}(R)$ $\forall\, n \in I$

& $f_n(z) \xrightarrow{\text{unif.}} f(z)$ in R

then, $f(z) \in \mathcal{C}(R)$

and

$$\lim_{n\to\infty} \int_C f_n(z)\,dz = \int_C f(z)\,dz$$

for any finite contour C in R.

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No proof req'd!

The ratio test is a corollary to this th^m.

Th^m (12.2) ("Weierstrass M-test")

Let $|b_j(z)| \le M_j$ in a region R w/ M_j constant.

If $\sum_{j=1}^{\infty} M_j$ converges ($< \infty$); then the series

$S(z) = \sum_{j=1}^{\infty} b_j(z)$ converges "uniformly" in R.

Examples & Applications

In this section we will consider 2 cases; the first is an example of a f^n sequence that does not conv. uniformly to its limit & the 2nd is an application of the Weierstrass M-test.

<u>eg ①</u> :- Sequence of partial sums comprising the geometric series.

Recall that $s_n(z) = \sum_{k=0}^{n} z^k \longrightarrow s(z) = \frac{1}{1-z}$

$\forall \; z \in D_1(0)$
(i.e. $|z| < 1$)

$\because \; \mathbb{R} \subset \mathbb{C}$
we restrict our analysis to $z \in \mathbb{R}$,
whence $D_1(0) \equiv (-1, 1)$

& $|s_n(z) - s(z)| < \epsilon$

$\equiv |s_n(x) - s(x)| < \epsilon$

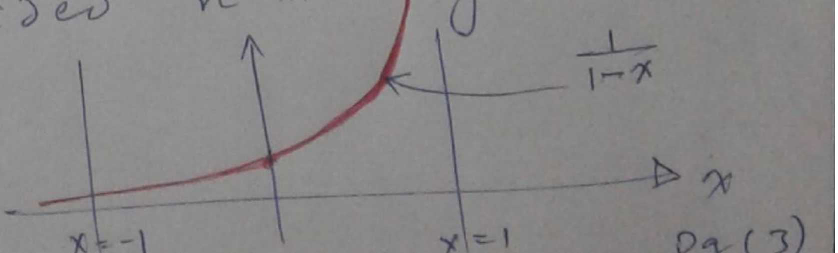
$\Rightarrow \; -\epsilon < s_n(x) - s(x) < \epsilon$

$\Rightarrow \; -\epsilon < s_n(x) < s(x) + \epsilon$

$\Rightarrow s(x) - \epsilon < s_n(x)$

This means if $s_n(x) \xrightarrow{\text{unif.}} s(x)$ then $s_n(x)$ is w/in an ϵ bandwidth of $s(x)$ $\forall \; x \in (-1, 1)$ provided n is large.

$\frac{1}{1-x}$ looks like



But $B_n(x) := |s_n(x) - s(x)|$

$$= \left| \sum_{k=0}^{n} x^k - \frac{1}{1-x} \right| = \left| \frac{\sum_{k=0}^{n}(x^k - x^{k+1}) - 1}{1-x} \right|$$

$$= \left| \frac{|x|^{n+1} - 1}{1-x} \right| \to \infty$$

as $x \to 1$ for fixed but large n.

$s_n(x) \xrightarrow{\text{unif}} s(x)$ on a compact subset of $(-1,1)$

We show this next!

eg (2) Application of Weierstrass M-test.

Let $[a,b]$ be a compact subset of $(-1,1)$.
Choose $q \in (0,1)$ s.t. $-1 < -q \le a < b \le q < 1$.
If $x \in [a,b] \Rightarrow |x| \le q$

$\Rightarrow |x^n| = |x|^n \le q^n$ ($\equiv M_n$ of Weierstrass M-test).

$$\sum_{n=0}^{\infty} q^n < \infty$$

$\Rightarrow \sum_{n=0}^{\infty} x^n$ converges uniformly in $[a,b]$ by the Weierstrass M-test.

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§(12.2) Taylor Series (review of basic ideas)

Power Series: $f(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$

If $z_0 = 0$

$$f(z) = \sum_{j=0}^{\infty} b_j (z)^j \quad \text{——— } (12.2.1)$$

Uniform convergence of this series.

Th^m (12.3) :- If the series in (12.2.1) converges for some $z_* \neq 0$, then it converges $\forall$ z in $|z| < |z_*|$

Moreover, it converges uniformly in $|z| \leq R$, for $R < |z_*|$.

Proof not req'd!

Th^m (12.4) (Taylor series for analytic f's & uniform convergence).

Let $f(z)$ be analytic for $|z - z_0| \leq R$.

then $f(z) = \sum_{j=0}^{\infty} b_j (z - z_0)^j$ where $b_j = \frac{f^{(j)}(z_0)}{j!}$,

converges "uniformly" in $|z - z_0| \leq R_1 < R$

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<u>proof</u> :- We will prove this thm for $z_0 = 0$ WLOG.

Cauchy integral formula from Lecture (11) states

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)} d\zeta$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta} \left(1 - \frac{z}{\zeta}\right)^{-1} d\zeta$$ where C is a circle of radius R

this is true b/c $\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$ is a uniformly convergent series $\forall |z| < 1$

Here, since z is interior to $C \Rightarrow \left|\frac{z}{\zeta}\right| < 1$.

$$= \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta} \sum_{j=0}^{\infty} \left(\frac{z}{\zeta}\right)^j d\zeta$$

Note that the conv. of this series was absolutely essential in this step. (else the integral become unbounded).

$$= \frac{1}{2\pi i} \oint_C f(\zeta) \sum_{j=0}^{\infty} \frac{z^j}{(\zeta)^{j+1}} d\zeta$$

$$\overset{A}{=} \sum_{j=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta)^{j+1}} d\zeta\right)}_{\text{call this } b_j} z^j$$

$$= \sum_{j=0}^{\infty} b_j z^j \; ; \text{ where } b_j = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{\zeta^{j+1}} d\zeta = \frac{f^{(j)}(0)}{j!}$$

Why? #

Q) Why is this step A valid?

Hint: $f^{(k)}(z) = \frac{k!}{2\pi i} \oint \frac{f(\zeta)\, d\zeta}{(\zeta - z)^{k+1}}$ from Cauchy int. formula.

Ans :- $s(z, \zeta) = \lim_{n \to \infty} \sum_{j=0}^{n} \frac{z^j}{\zeta^{j+1}} = \lim_{n \to \infty} s_n(z, \zeta)$ & then apply thm (12.1). Pg (6) example of th^m (12.4)

Find the taylor series representation of $f(z) = e^{z^2}$.

Ans:- We will first consider

$$\tilde{f}(z) = e^z \text{ which is analytic in } \mathbb{C}.$$

$\Rightarrow \exists$ a taylor series form.

$$\tilde{f}(z) = e^z = \sum_{j=0}^{\infty} b_j z^j$$

$$\text{where } b_j = \frac{f^{(j)}(0)}{j!}$$

$$= \frac{1}{j!} \quad b/c \; e^0 = 1$$

infinite R.O.C

This is called Hadamard's formula.

$$\therefore \tilde{f}(z) = e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} \quad \forall \; |z| < \infty.$$

b/c Ratio test implies

$$\lim_{n\to\infty} \left| \frac{z^{n+1}/(n+1)!}{z^n/n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{z}{n+1} \right|$$

$$= 0 \; \#$$

Replace z by z^2 to get $f(z) = e^{z^2} = \sum_{j=0}^{\infty} \frac{z^{2j}}{j!}$

Radius of Convergence (R.O.C)

The largest no. R for which the power series in th^m (12.4) converges inside the disk $|z| < R$ is called R.O.C. (R may be $0, \infty$ or finite non-zero)

Alternatively, $R := \left(\lim_{n\to\infty} \sup_{m\geq n} |a_m|^{\frac{1}{m}}\right)^{-1}$.

* Termwise integration & differentiation of Taylor series is valid (with uniform convergence holding in each case).

* product of 2 convergent series.

$$\underbrace{f(z)}_{\sum_j a_j z^j}\underbrace{g(z)}_{\sum_j b_j z^j} = \sum_{j=0}^{\infty} c_j z^j$$

where $c_j = \sum_{k=0}^{j} b_k a_{j-k}$

* the comparison test (as w/ the reals) also applies in $\mathbb{C}$.

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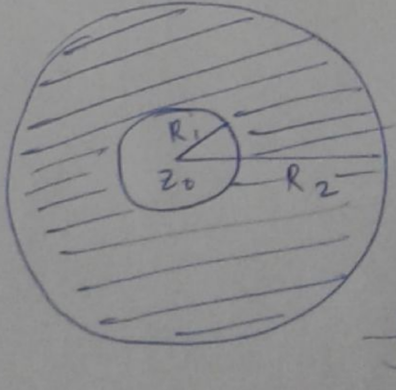
§ (12.3) Laurent Series

Q) Why do we need a new type of power Series when we have our famous Taylor series?

Ans :- In many applications we encounter f^n's that are "not" analytic at some pts or in some regions of the complex plane & hence Taylor expansions cannot be employed in the neighborhood of such points. Laurent series is often the answer.

Laurent series involves both +ve and -ve powers of $(z - z_0)$. Such a series is valid for those f^n's that are analytic in & on a circular annulus $R_1 \leq |z - z_0| \leq R_2$.



Shaded region is region of analyticity of $f(z)$; hence $f(z)$ has a valid Laurent series.

Th^m (12.5) (Laurent Series & Unif. convergence)

A f^n $f(z)$ which is analytic in an annulus $R_1 \le |z - z_0| \le R_2$ may be represented by the power series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad \text{—(12.3.1)}$$

in the region $R_1 < R_a \le |z - z_0| \le R_b < R_2$,

$$\text{Where } c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz, \quad \text{—(12.3.2)}$$

C is a Jordan contour in the region of analyticity enclosing the inner bdy $|z - z_0| = R_1$

Moreover, The Laurent series of $f(z)$ given by (12.3.1) & (12.3.2) in the annulus mentioned above <u>converges "uniformly"</u> to $f(z)$ for $\rho_1 \le |z - z_0| \le \rho_2$ where $R_1 < \rho_1$ & $\rho_2 < R_2$. #

<u>Proof</u>:- We will present a proof of this important Th^m in the next lecture!

* Often, while writing the Laurent series expansion of a f^n. $f(z)$; we "<u>rarely</u>" use eq (12.3.2) to find the coefficients c_n. Instead the coeff. follow naturally by other considerations as will be demonstrated in the examples that follow.

Some important notes about Laurent Series:

* Residue of $f(z)$ $= C_{-1}$ (i.e. the coeff. of $\frac{1}{z-z_0}$)

$$= \frac{1}{2\pi i} \oint_C f(z)\,dz$$

* Principal part of $f(z)$ = the $-$ve powers of the Laurent series.

* Laurent Series $\xrightarrow{\text{conv. to}}$ Taylor Series if $f(z)$ is analytic inside $|z-z_0| = R_1$ b/c by Cauchy's th^m $C_n = 0 \ \forall\ n \leq -1$.

* Laurent Series $\longrightarrow \sum_{n=-\infty}^{0} C_n (z-z_0)^n$ if $f(z)$ is analytic outside the circle $|z-z_0| = R_2$

This can be shown by substituting $t = \frac{1}{z}$.

* Laurent Series is a unique power series.

§ (12.3.1) Examples of Laurent Series:

eg (12.3.1(a)) :- Find the Laurent series of $f(z) = \frac{1}{1+z}$ for $|z| > 1$.

Soln:- We know by Taylor series:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for } |z| < 1. \qquad (12.3.1)$$

Now $\frac{1}{1+z} = \frac{1}{z(1+\frac{1}{z})} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}$ (replace z by $-1/z$ in eq(12.3.1)) & this is legit b/c $|z|>1 \Rightarrow |-\frac{1}{z}|<1$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}$$

$$\frac{1}{1+z} = \underbrace{\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots}_{\text{Laurent series of } \frac{1}{1+z}\ \forall |z|>1.}$$

Also for $|z|<1$, $\frac{1}{1+z} = \sum_{n=0}^{\infty}(-1)^n z^n$.

Thus there are different ~~reg~~ series expansions of $\frac{1}{1+z}$ in different regions of the complex plane,

i.e. $$\frac{1}{1+z} = \begin{cases} \sum_{n=0}^{\infty}(-1)^n z^n \ ; & |z|<1 \\ \sum_{n=0}^{\infty}(-1)^n z^{-(n+1)} \ ; & |z|>1 \end{cases}$$

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example (12.3.1(b))

Q) Find the Laurent expansion of

$$f(z) = \frac{1}{(z-1)(z-2)} \quad \text{for } 1<|z|<2$$

Soln:- Using method of partial fractions

$$f(z) = -\frac{1}{(z-1)} + \frac{1}{(z-2)}$$

By taking a cue from the previous example, we write f(z) in this form

$$= -\frac{1}{z}\left(\frac{1}{1-\frac{1}{z}}\right) - \frac{1}{2}\left(\frac{1}{1-z/2}\right)$$

$$= -\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} \quad -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$

for $1<|z|<2$ i.e. $\left|\frac{1}{z}\right|<1$

for $1<|z|<2$ $|z/2|<1$

$$= -\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right)$$

$$\therefore f(z) = \frac{1}{(z-1)(z-2)} = \sum_{n=-\infty}^{\infty} c_n z^n \text{ where}$$

$$c_n = \begin{cases} -1 & ; n \le -1 \\ \frac{1}{2^{n+1}} & ; n \ge 0 \end{cases}$$

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