§ (8.4)

Corollary of Monotone Seq. Thm.

$\sum_{n=1}^{\infty} a_n$ converges $\iff$ its partial sums are bdd from above.

eg: $\sum a_n \equiv \sum \frac{1}{n}$ div. (harmonic series div.

b/c seq of its partial sums is not bdd

(If $n = 2^k$, $s_n > k/2$ . . . . $\Rightarrow s_n$ is not bdd)

The Integral Test (for convergence).

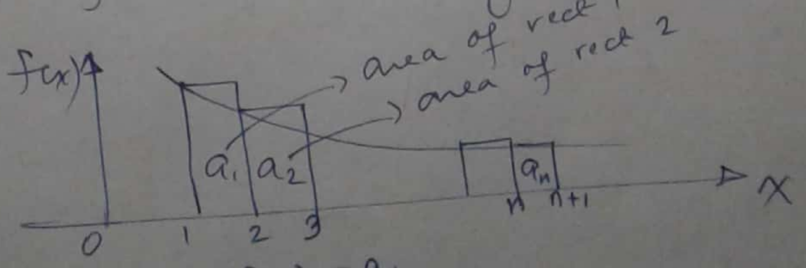
Let $\{a_n\}$ be a sequence of positive terms.

Let $a_n = f(n)$; f is continuous, +ve, dec. f^n of x $\forall x \geq N$ $(N \in I^+)$.

then $\sum_{n=N}^{\infty} a_n$ & $\int_N^{\infty} f(x)dx$ converge/diverge alike, together.

Proof :- We discuss the case for $N = 1$

Let f be decreasing w/ $f(n) = a_n$ $\forall n$.

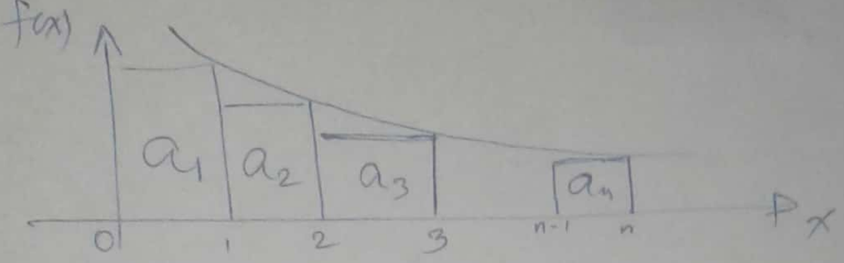


(1) $f(1) = a_1$

$f(2) = a_2$

⋮

By observation, $a_1 + a_2 + \cdots + a_n \geq \int_1^{n+1} f(x)dx$ —— ①



$$a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,dx \quad \text{—— (2)}$$

(1) & (2) $\Rightarrow \int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \cdots + a_n \le a_1 + \int_1^n f(x)\,dx$ —— (3)

As $n \to \infty$,

$\int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} a_n \Rightarrow$ if $\int_{N=1}^{\infty} f(x)\,dx$ diverges (∞)

then $\sum_{n=1}^{\infty} a_n$ div.

& $\sum_{n=1}^{\infty} a_n \le a_1 + \int_1^{\infty} f(x)\,dx \Rightarrow$ If $\int_1^{\infty} f(x)\,dx$ div

$\Rightarrow \sum_{n=1}^{\infty} a_n$ div.

$\therefore \int_{N=1}^{\infty} f(x)\,dx$ & $\sum_{n=1}^{\infty} a_n$ div & conv. alike (together).

The same arguments can be generalized for any $N \ge 1$.

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eg (Application of the ~~p-series~~ integral test).

Show:- $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$

$p \in \mathbb{R}$ (Constant)

$\begin{cases} \text{Converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$

Soln:- (i) If $p > 1$; $f(x) = \frac{1}{x^p}$ is a +ve dec. f^n of x.

$$\therefore \int_1^{\infty} \frac{1}{x^p}\,dx = \lim_{\alpha \to \infty} \int_1^{\alpha} \frac{1}{x^p}\,dx = \lim_{\alpha \to \infty} \left(\frac{x^{-p+1}}{-p+1}\right)_1^{\alpha}$$

$$= \left(\frac{1}{1-p}\right) \lim_{\alpha \to \infty} \left(\frac{1}{\alpha^{p-1}} - 1\right)$$

$$\overset{b/c\ p-1>0}{=} \left(\frac{1}{1-p}\right)(0-1) = \frac{1}{p-1} < \infty$$

$\therefore$ the series $\sum_{p=1}^{\infty} \frac{1}{n^p}$ converges by the integral test.

(ii) If $p < 1$; $\Rightarrow 1-p > 0$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \frac{1}{1-p} \lim_{\alpha \to \infty} (\alpha^{1-p} - 1) = \infty$$

$\Rightarrow \sum_{p=1}^{\infty} \frac{1}{n^p}$ diverges by integral test.

(iii) If $p = 1$; we have the Divergent harmonic series. #

Application of integral test.

eg. ① Test the series for convergence / divergence
$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

Soln:- Note $f(x) = \frac{1}{x^2+1} > 0$, continuous & decreasing on $[1, \infty)$

$$\int_1^{\infty} \frac{1}{x^2+1}\, dx = \lim_{\alpha \to \infty} \int_1^{\alpha} \frac{1}{x^2+1}\, dx = \lim_{\alpha \to \infty} \tan^{-1} x \Big|_1^{\alpha}$$
$$= \lim_{\alpha \to \infty} \left(\tan^{-1}\alpha - \frac{\pi}{4}\right)$$
$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} < \infty$$

$\Rightarrow$ By the integral test that $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent.

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eg ② Determine whether the series $\sum_{n=1}^{\infty} \frac{\log n}{n}$ converges / diverges.

Soln:- Consider $f(x) = \frac{\log x}{x} > 0$, continuous $\forall$ $x > 1$ b/c ratio of 2 continuous f^n is continuous

Also $f'(x) = \frac{1}{x^2} - \frac{\log x}{x^2} = \frac{1 - \log x}{x^2} < 0$

when $\log x > 1$ i.e. when $x > e$

$\Rightarrow$ $f(x)$ is decreasing when $x > e$.

So now we apply the integral test

$$\int_1^\infty \frac{\log x}{x} = \lim_{\alpha\to\infty}\int_1^\alpha \frac{\log x}{x}\,dx = \lim_{\alpha\to\infty} \left.\frac{(\log x)^2}{2}\right|_1^\alpha$$

$$= \lim_{\alpha\to\infty} \frac{(\log \alpha)^2}{2} - 0$$

$$= \infty$$

$\therefore \sum_{n=1}^{\infty} \frac{\log n}{n}$ is divergent.

Also note that the requirement $f(x)$ be decreasing $\forall x$ is not necessary, what is important is that $f(x)$ is "eventually" decreasing! (i.e. dec. for all $x > N \in \mathbb{R}^+$).

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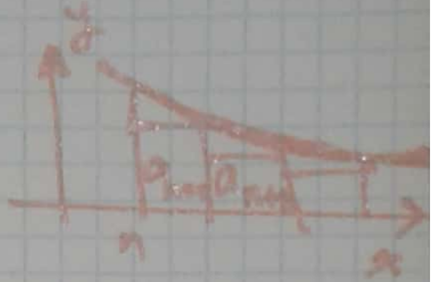
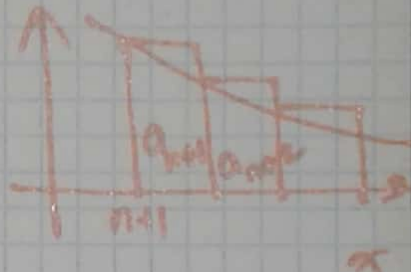
Bounds for the error of approximation of a series whose ~~[illegible]~~ sum $s \simeq s_n$.

Let us say we know that $\sum_{n=1}^{\infty} a_n$ is convergent by the integral test. We may now want to know that what exactly is the sum of the series. Of course we may resort to s_n (the seq. of partial sums) b/c $s_n \longrightarrow s$.

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \cdots$$

Following the ideas from the proof of the integral test, we may infer:-





$$R_n = a_{n+1} + a_{n+2} + \cdots \leq \int_n^\infty f(x)\,dx \quad \text{— (1)}$$

$$R_n = a_{n+1} + a_{n+2} + \cdots \geq \int_{n+1}^\infty f(x)\,dx \quad \text{— (2)}$$

(I) & (II) $\Rightarrow$

$$\boxed{\int_{n+1}^\infty f(x)\,dx \leq R_n \leq \int_n^\infty f(x)\,dx}$$

Also note

$$\boxed{s_n + \int_{n+1}^\infty f(x)\,dx \leq R_n + s_n = s \leq s_n + \int_n^\infty f(x)\,dx}$$

Thus we obtain a lower & upper bound for the sum of a convergent series $s = \sum_{n=1}^\infty a_n$.

Are these bounds for s useful ??

Let us try to answer this by considering the following example.

eg) How many terms are required to ensure that the sum $\sum \frac{1}{n^3}$ is accurate to w/in 0.0005?

Soln:- Consider $f(x) = \frac{1}{x^3} > 0$, decreasing & $C[1,\infty)$

$$\text{so } \int_n^\infty \frac{1}{x^3}\,dx = \lim_{\alpha\to\infty}\left(-\frac{1}{2x^2}\right)_n^\alpha = \cdots = \frac{1}{2n^2}$$

Accuracy to w/in 0.0005 means we need to find n s.t. $R_n \leq 0.0005$

$$\because R_n \leq \int_n^\infty \frac{1}{x^3}\,dx = \frac{1}{2n^2} < 0.0005$$

$$\Rightarrow n^2 > \frac{1}{0.001} = 1000 \Rightarrow n > \sqrt{1000} \approx 31.6$$

i.e. We need 32 terms to ensure an accuracy w/in 0.0005.

But is it really the case ?? Can we do something smarter?

What if we use the 2^{nd} set of bounds w/ only $n = 10$ terms i.e. $s_{10} + \int_{n+1}^{\infty} \frac{1}{x^3}\,dx \le s \le s_{10} + \int_{n}^{\infty} \frac{dx}{x^3}$

$\Rightarrow s_{10} + \frac{1}{2(11)^2} \le s \le s_{10} + \frac{1}{2(10)^2}$

Using $s_{10} \approx 1.197532$ (Check it w/ your calculator)

$1.201664 \le s \le 1.202532$ —— (I)

If we $s \approx \frac{1.201664 + 1.202532}{2}$

$= 1.2021$

then error is at most (at max) $\frac{\left(1.202532 - 1.201664\right)}{2}$

$= 0.000434$

< 0.0005

b/c the farthest s can be from the actual sum is half the length of the diff. of the b/ds if the actual sum is actually the L.H.S./R.H.S. of the ineq (I).

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