Computational solutions to systems of linear equations

Shall we now crank the wheel and churn some numbers in the computer? Shall we learn how to do math with a computer?

eg. 1) Let us consider the following system of equations:

$$x_1 - x_2 + x_3 + x_4 = 2 \text{..........}(i)$$
$$x_1 + x_2 + x_3 - x_4 = 3 \text{..........}(ii)$$
$$x_1 + 3x_2 + x_3 - 3x_4 = 1 \text{..........}(iii)$$

Does this set of equations have a solution?

Let's perform the following operations to eliminate x_1:
eq. (v) : $(ii) - (i)$ and eq. (vi) : $(iii) - (i)$.

$$x_1 - x_2 + x_3 + x_4 = 2 \text{..........}(iv)$$
$$x_2 - x_4 = \frac{1}{2} \text{. . . . .}(v)$$
$$4x_2 - 4x_4 = -1 \text{..........}(vi)$$

Next we will attempt to eliminate x_2 from $eq.\ (vi)$: eq. (ix) : $(vi) - 4(v)$,

$$x_1 - x_2 + x_3 + x_4 = 2 \text{..........}(vii)$$
$$x_2 - x_4 = \frac{1}{2} \text{. . . . .}(viii)$$
$$0 = -3 \text{..........}(ix) \qquad \text{OOPS!}$$

*This is a contradiction and hence the above system of equations is **inconsistent** (no solutions)!*[6]

[6] This should have been obvious by inspection of the original system of equations because there were 4 unknowns and only 3 equations!

eg. 2) Now consider the following system of equations:

$$x_1 + 4x_2 + 2x_3 = -2 \dots\dots\dots\dots\dots\dots (i)$$
$$-2x_1 - 8x_2 + 3x_3 = 32 \dots\dots\dots\dots\dots\dots (ii)$$
$$x_2 + x_3 = 1 \dots\dots\dots\dots\dots\dots (iii)$$

Does the above system of equations have a solution?

$Eq.(ii): \quad (ii) + 2(i)$ gives

$$x_1 + 4x_2 + 2x_3 = -2 \dots\dots\dots\dots\dots\dots (iv)$$
$$7x_3 = 28 \dots\dots\dots\dots\dots\dots (v)$$
$$x_2 + x_3 = 1 \dots\dots\dots\dots\dots\dots (vi)$$

Swap $eq.(v)$ and (vi):

$$x_1 + 4x_2 + 2x_3 = -2 \dots\dots\dots\dots\dots\dots (vii)$$
$$x_2 + x_3 = 1 \dots\dots\dots\dots\dots\dots (viii)$$
$$7x_3 = 28 \dots\dots\dots\dots\dots\dots (ix)$$

Eq. $(xii): \quad \frac{1}{7}(ix)$:

$$x_1 + 4x_2 + 2x_3 = -2 \dots\dots\dots\dots\dots\dots (x)$$
$$x_2 + x_3 = 1 \dots\dots\dots\dots\dots\dots (xi)$$
$$x_3 = 4 \dots\dots\dots\dots\dots\dots (xii)$$

Now that we know $x_3 = 4$, we *back substitute* the knowns and compute the remaining unknowns: $x_2 = 1 - x_3 = -3$ and $x_1 = -2 - 4x_2 - 2x_3 = 2$. Therefore, this system of equation has a unique solution. This example will be very similar to the numerical technique we will learn in this section known as *Gauss elimination.*

Before we study this new method (Gauss elimination), let us look at one more example here below.

eg. 3) Consider the following system of equations:

$$x_1 + 3x_2 + 3x_3 + 2x_4 = 1 \; \dots\dots\dots\dots\dots\dots (i)$$
$$2x_1 + 6x_2 + 9x_3 + 5x_4 = 5 \; \dots\dots\dots\dots\dots\dots (ii)$$
$$-x_1 - 3x_2 + 3x_3 = 5 \; \dots\dots\dots\dots\dots\dots (iii)$$

$Eq.\,(ii): \quad (ii) - 2(i)$ and $(iii): \quad (iii) + (i)$

$$x_1 + 3x_2 + 3x_3 + 2x_4 = 1 \; \dots\dots\dots\dots\dots\dots (i)$$
$$3x_3 + x_4 = 3 \; \dots\dots\dots\dots\dots\dots (ii)$$
$$6x_3 + 2x_4 = 6 \; \dots\dots\dots\dots\dots\dots (iii)$$

Note x_2 has disappeared from eqs. (ii) and (iii); so we proceed to the next unknown x_3! Eq. $(ii): \; \frac{1}{3}(ii)$ followed by $eq.\,(iii): \quad (iii) - 6(ii)$:

$$x_1 + 3x_2 + 3x_3 + 2x_4 = 1 \; \dots\dots\dots\dots\dots\dots (i)$$
$$x_3 + \frac{1}{3}x_4 = 1 \; \dots\dots\dots\dots\dots\dots (ii)$$
$$0 = 0 \; \dots\dots\dots\dots\dots\dots (iii)$$

Here the third equation tells us nothing and can be ignored. x_4 and x_2 can be assigned arbitrary values: $x_4 = c, \quad x_2 = d;$ to recover (by back substitution) $x_1 = -2 - c - 3d; \quad x_3 = 1 - c/3.$

This system of linear equations has *<u>infinitely</u>* *many solutions!*

What have we learnt through these examples?

I claim we have learnt (perhaps unknowingly) the following:

- **the operational mechanism of Gauss elimination,**
- **echelon form, pivots, etc …**

Gauss Elimination method *(direct computational method!)*:

I finish in $O(n^3)$ steps!

We are solving system of linear equations of the form:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \;\ldots\ldots\ldots\ldots (i) \\
a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \;\ldots\ldots\ldots\ldots (ii) \\
\cdot \qquad\qquad \cdot &= \cdot \\
\cdot \qquad\qquad \cdot &= \cdot \\
\cdot \qquad\qquad \cdot &= \cdot \\
a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n &= b_m \;\ldots\ldots\ldots\ldots (\text{m}^{th} \text{ eq.})
\end{aligned}
$$

<u>Procedure</u>:

i) Find an equation in which x_1 appears and, if necessary, interchange this equation with the first equation. Thus we can assume that x_1 appears in the first equation.

ii) Multiply eq. (i) by a suitable non-zero scalar in such a way as to make the coefficient of x_1 equal to 1.

iii) Subtract suitable multiples of eq. (i) from eqs. (ii) through (m) in order to eliminate x_1 from these equations.

iv) Inspect equations (ii) through (m) and find the first equation which involves one of the unknowns $x_2, \ldots, x_n$, say x_{i_2}. By interchanging equations once again, we can suppose that x_{i_2} appears in eq. (ii).

v) Multiply eq. (ii) by a suitable non-zero scalar to make the coefficient of x_{i_2} equal to 1.

vi) Subtract multiples of eq. (ii) from eq. (iii) through (m) to eliminate x_{i_2} from these equations.

$$\begin{aligned}
x_{i_1} + *x_{i_2} + \cdots * x_n &= * \;\ldots\ldots\ldots\ldots (i)\\
x_2 + \cdots * x_n &= * \;\ldots\ldots\ldots\ldots (ii)\\
\cdot \qquad\quad \cdot &= \cdot\\
\cdot \qquad\quad \cdot &= \cdot\\
\cdot \qquad\quad \cdot &= \cdot\\
x_{i_r} + \cdots * x_n &= * \;\ldots\ldots\ldots\ldots (\text{r}^{th} \text{ eq.})\\
0 &= *\\
\cdot &= *\\
\cdot &= *\\
0 &= * \;\ldots\ldots\ldots\ldots (\text{m}^{th} \text{ eq.})
\end{aligned}$$

vii) Examine eps. (iii) through (m) and find the first one that involves an unknown other than x_1 and x_{i_2}, say x_{i_3}. Interchange equations so that x_{i_3} appears in eq. (iii).

This elimination procedure continues in this manner producing the so called <u>*pivotal*</u> *unknowns* $x_1 = x_{i_1}, x_{i_2}, \cdots, x_{i_r}$ *until we reach a linear system in which no further unknowns occur in the equations beyond the* r^{th} *equation. A linear system of this sort is said to be in* <u>*echelon form*</u>*.*

The i_j are integers which satisfy $1 = i_1 < i_2 < \cdots < i_r \leq n$. After arriving at the echelon form, we use <u>back substitution</u> to solve for the unknowns $x_1, x_2, \ldots, x_n$.

Q) What can be said about the solution(s) of the linear system by inspecting the echelon form?

Theorem:

(i) A linear system is <u>consistent</u> if and only if all the entries on the right hand sides of those equations in echelon form which contain no unknowns are zero.

(ii) If the system is consistent, the non-pivotal unknowns can be given arbitrary values; the general solution is then obtained by using back-substitution to solve for the pivotal unknowns.

(iii) The system has a unique solution if and only if all the unknowns are pivotal.

Matrix form of Gauss elimination

(i) $R_i \leftrightarrow R_j$

(ii) $R_i : \quad R_i + cR_j$

(iii) $R_i : \quad cR_i$

The matrix in row-echelon form will have a descending staircase structure:

$$\begin{pmatrix} 0 & \cdot & \cdot & 0 & 1 & * & \cdot & \cdot & * & \cdot & * & \cdot & \cdot & * & * \\ 0 & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & & 0 & 1 & \cdot & * & \cdot & \cdot & * \\ 0 & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 & \cdot & 0 & 1 & \cdot & \cdot & \cdot & * \\ 0 & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 & \cdot & 0 & \cdot & 0 & 1 & \cdot & * \\ 0 & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 & \cdot & 0 & \cdot & 0 & 0 & \cdot & * \\ 0 & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & 0 & \cdot & 0 & \cdot & 0 & 0 & \cdot & * \end{pmatrix}$$

eg. 1) So for example, **eg. 3** of the previous section can be solved by considering the matrix-vector form $A\mathbf{x} = \mathbf{b}$ and reducing the *augmented matrix* $\tilde{A} = \begin{pmatrix} A & | & \mathbf{b} \end{pmatrix}$ into the row-echelon form followed by back-substitution. The augmented matrix for **eg. 3** is:

$$\begin{pmatrix} 1 & 3 & 3 & 2 & | & 1 \\ 2 & 6 & 9 & 5 & | & 5 \\ -1 & -3 & 3 & 0 & | & 5 \end{pmatrix}.$$

The row echelon form is: $rref(\tilde{A}) = \begin{pmatrix} 1 & 3 & 3 & 2 & | & 1 \\ 0 & 0 & 1 & 1/3 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$. The non-pivotal entries are given arbitrary values $x_2 = d$ and $x_4 = c$. The pivotal entries are then calculated as $x_1 = -2 - c - 3d$ and $x_3 = 1 - c/3$.

Reading Assignments: *We will learn more about the following through exercise problems.*

(i) Practical challenges in Gauss elimination:

Pivoting: https://235d9ee8-8e8c-4d7b-a842-264ad94cf102.filesusr.com/ugd/334434_d627cffea18f4f81b195c15b37ec990e.pdf

Partial pivoting: https://235d9ee8-8e8c-4d7b-a842-264ad94cf102.filesusr.com/ugd/334434_e58e6bc8907f486b9de1253d60c52e3a.pdf

(ii) **Arithmetic complexity** of Gauss elimination: https://235d9ee8-8e8c-4d7b-a842-264ad94cf102.filesusr.com/ugd/334434_5a3eab64a8b0442cabd729aa5defab45.pdf

(iii) **Gauss Jordan elimination**: (similar but relies on reduced row echelon form instead of simply row echelon form that is used I Gauss elimination)

(iv) **LU decomposition**: https://235d9ee8-8e8c-4d7b-a842-264ad94cf102.filesusr.com/ugd/334434_5a3eab64a8b0442cabd729aa5defab45.pdf