Lecture (7) : Riemann Surfaces.

2/2/2019.

§(7.1)
By Riemann surface we mean an extension of the ordinary complex plane to a surface that has more than one "sheet". The multivalued f^n will have only one value corresponding to each point on the Riemann surface.

eg. Reconsider $\omega = \sqrt{z}$

Now, consider the two-sheeted surface as below.

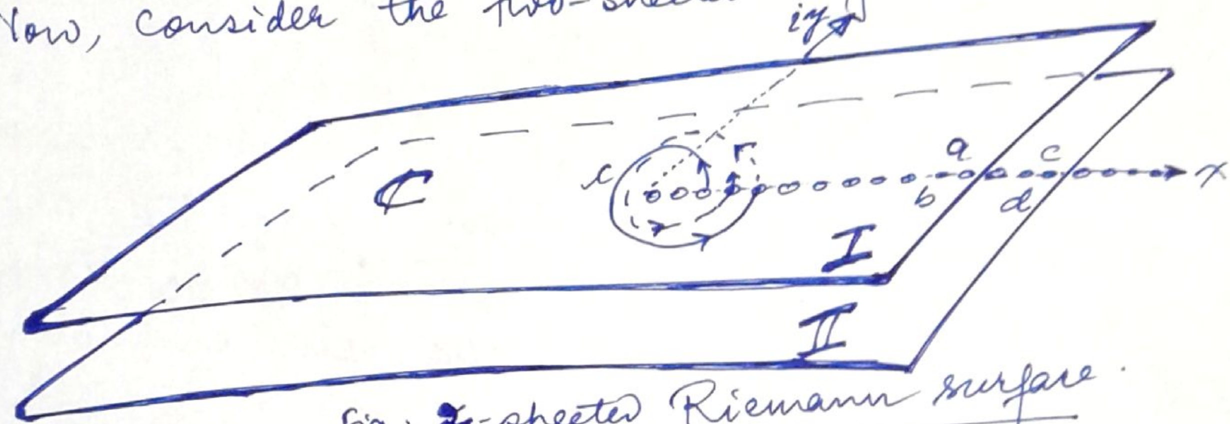

Fig: 2-sheeted Riemann surface.

Above, we have double copies I and II of the complex z-plane w/ a cut along the +ve x-axis.

Along the cut plane we have the planes (sheets) joined in the following manner:-

Cut along Ib ~~stitch~~ cut along IIc

Cut along Ia ~~stitch~~ cut along IId

In this way, we produce a continuous one-to-one map from the Riemann surface for the f^n. $z^{1/2}$ onto the ω-plane $\omega = u + iv = z^{1/2}$

If we follow the curve c, we begin on sheet Ia - wind around the origin (B.P.) to Ib, We then squeeze through the cut & come out on IIc; We again wind around the origin to IId,

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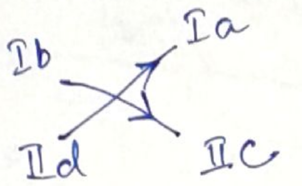
go through the cut & come out on Ia.
then repeat.

Note :- there is "no" ambiguity w/ any intersecting edge of the stitches/glue.
To convince yourself of this consider a side view from the right.

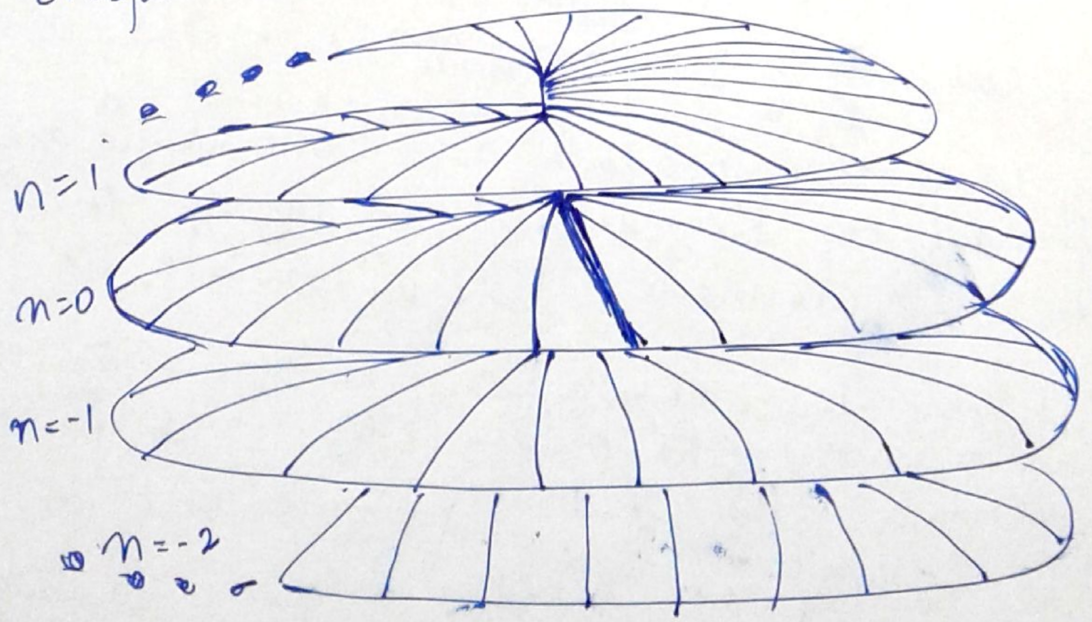


The arrows shown here are the side views of the curve c. clearly due to the uniquely different directions of ~~the~~ curve along the "joints" there is no ambiguous "short circuit".

§(7.2) Riemann Surface for infinitely multi-valued f^n.

eg. $\log z = \log|z| + i(\theta_p + 2n\pi)$; $0 \le \theta_p < 2\pi$

Consider the following ∞ly sheeted Riemann surface.



We will conclude our discussion on multivalued functions by studying 2 useful Th^{m}s, the second of which gives us an elegant method of calculating the derivative of an analytic f^n.

§(7.3)

Def^n (univalence) :- $f(z)$ is univalent in a domain G if it is one-to-one & analytic in G.

G is then called the domain of univalence for $f(z)$.

* If $f(z)$ is univalent in $G \Rightarrow f'(z) \neq 0$ in G.

Th^m (7.3.1) Let $w = f(z) = u(x,y) + iv(x,y)$ ① be univalent in G, and let E be the image of G under the mapping ① then E is also a domain in the w-plane.

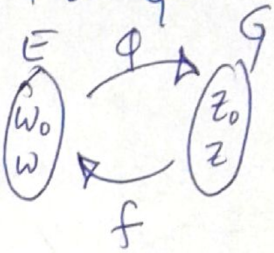
* We will skip the proof of Th^m (7.3.1) but it should be intuitively believable.

$$z \underset{\longleftarrow \varphi}{\overset{\longrightarrow f}{\longrightarrow}} w$$

Th^m (7.3.2) Let $w = f(z)$, G and E be the same as above. Further, let $z = \varphi(w)$ be the inverse of $w = f(z)$. Then $\varphi(w)$ is univalent in E w/ derivative $\varphi'(w) = \dfrac{1}{f'(z)}$

Proof (7.3.2) :-

$z = \varphi(w)$ is obviously single-valued & one-to-one in E, ~~f~~ b/c $w = f(z)$ is 1 to 1 in G.



Let $w_0, w \in E \overset{\rightarrow \varphi}{\underset{\leftarrow f}{\longleftrightarrow}} z_0, z \in G$

$\varphi(w) = x(u,v) + i y(u,v) \in C(E)$ b/c

$$\left.\begin{aligned} Re(\varphi) &= x(u,v) \\ Im(\varphi) &= y(u,v) \end{aligned}\right\} \text{both Continuous!}$$

Why ??

(this is technical & a digression here but you can try it yourself)

$\Rightarrow z \to z_0$ as $w \to w_0$

i.e. $\lim_{w \to w_0} z = \lim_{w \to w_0} \varphi(w) \xrightarrow[\text{cont.}]{\varphi \text{ is}} \varphi(w_0) = z_0$

$$\text{Now } \varphi'(w_0) = \lim_{w \to w_0} \frac{\varphi(w) - \varphi(w_0)}{w - w_0}$$

$$= \lim_{w \to w_0} \frac{z - z_0}{w - w_0} = \lim_{z \to z_0} \frac{1}{\frac{w - w_0}{z - z_0}}$$

$$= \lim_{z \to z_0} \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}$$

$$= \frac{1}{f'(z_0)}$$

this is true for all w_0, z_0 in G and E

$$\therefore \varphi'(w) = \frac{1}{f'(z)}$$

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Application of Th^m (7.3.2) to find derivative of an analytic f^n (in the cut plane).

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eg. ~~exp~~ $z = e^{w}$; $w = u + iv$

Let us reconsider the exponential f^n in $\mathbb{C}$.
Our goal is to obtain $\frac{d}{dz}\log w$ (or equivalently $\frac{d}{dz}\log z$).

To find this, we will use Th^m (7.3.2)

First we will show that $z = e^{w}$ is univalent in a certain $E_{\mathbb{C}}$ (yet to be found).

Let $w_1 = u_1 + iv_1$ and $w_2 = u_2 + iv_2$ be 2 pts on w-plane.

e^{w} is univalent unless $\exists$ a $w_1 \neq w_2$ for which $e^{w_1} = e^{w_2}$.

If $u_1 \neq u_2$; then e^{w} is surely univalent.

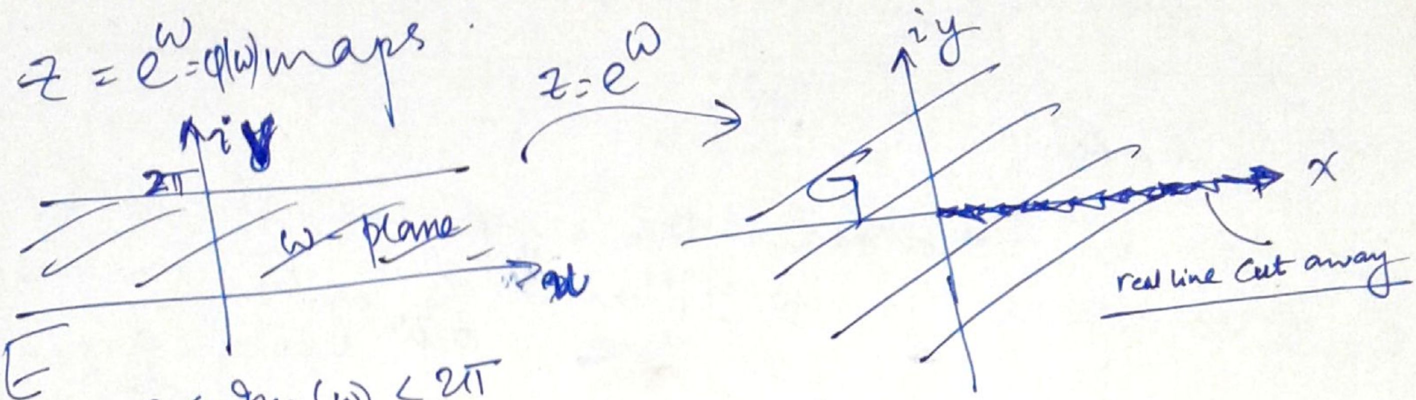
If $u_1 = u_2 = u$ then
$$e^{w_1} - e^{w_2} = e^{u}(e^{+iv_1} - e^{+iv_2})$$
$$= e^{u}(\cos v_1 + i\sin v_1 - \cos v_2 - i\sin v_2)$$
$$= e^{u}(\cos v_1 - \cos v_2 + i(\sin v_1 - \sin v_2))$$
$$= 2i\sin\left(\frac{v_1 - v_2}{2}\right)e^{u}e^{i\frac{(v_1+v_2)}{2}}$$
$= 0$ only when $(v_1 - v_2) = 2n\pi$; $n \in \mathbb{Z}$

i.e. e^{w} is univalent unless $v_1 - v_2 = 2n\pi$; $n \in \mathbb{Z}$

or e^{w} is univalent in any strip of the form $C < \operatorname{Im}(w) < C + 2\pi$; $C \in \mathbb{R}$ const.

Set $c = 0$.

$z = e^{w} = \varphi(w)$ maps



$0 < \operatorname{Im}(w) < 2\pi$

$$z = e^{w} = e^{u} e^{iv}$$

$v = 0 \Rightarrow z = e^{u} = x + iy$. Comparing real & Img. parts.

is not included $\Rightarrow y = 0$. should not be included (Branch cut).

$v = 2\pi \Rightarrow z = e^{u} e^{i2\pi}$
$= e^{u} = x + iy$

$\Rightarrow$ again maps to $y = 0$.

Note in each of above case

$\underline{\underline{u > 0}}$

$e^{u} = x > 0$.

$\underline{\underline{u < 0}} \Rightarrow -|u| = u$

$e^{u} = e^{-|u|} = \frac{1}{e^{|u|}} = x > 0$ $\Big\} \Rightarrow$ It's not the entire real axis but merely the +ve real axis that is cut (Branch cut)

$\therefore \varphi(w) = e^{w}$ is univalent in E $(0 < \operatorname{Im}(w) < 2\pi)$

$\Rightarrow$ inv. f^n $w = f(z) = \log z$ is univalent in G ($\mathbb{C}$ − {+ve real axis}).

by Th^m (7.3.2).

& $f'(z) = \frac{d}{dz} \log z = \frac{1}{\varphi'(w)} = \frac{1}{e^{w}} = \frac{1}{z}$.