Field

What is our playing field? What are we playing with?

Definition (Field): A field is a set $\mathbb{F}$ of numbers[1] with the property that if $a, b \in \mathbb{F}$, then $a + b$, $a - b$, ab and $\frac{a}{b}$ are also in $\mathbb{F}$ (assuming, of course, that $b \neq 0$ in the expression $\frac{a}{b}$. eg. $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are fields of numbers. $\mathbb{N}$ and $\mathbb{Z}$ are **not** fields of numbers!

Vector Spaces[2]

Which space are we talking about? What is it made up of? What can we play in this space?

Definition (Vector space): A vector space, $\mathscr{V}$ consists of a set $\mathbb{V}$ of vectors, a field $\mathbb{F}$ of scalars, and two operations:

i. *vector addition: if $v, w \in \mathbb{V}$, then $v + w \in \mathbb{V}$,*

ii. *scalar multiplication: $c \in \mathbb{F}$ and $v \in \mathbb{V}$ produces a new vector $cv \in \mathbb{V}$.*

These scalars and vectors satisfy the following *axioms.*

i. *Associativity of addition: $(v + u) + w = v + (u + w) \quad \forall v, u, w \in \mathbb{V}$.*

ii. Associativity *of multiplication: $(ab)u = a(bu)$, for any $a, b \in \mathbb{F}$, $u \in \mathbb{V}$.*

iii. *Distributivity: $(a + b)u = au + bu$ and $a(u + v) = au + av \quad \forall a, b \in \mathbb{F}, u \in \mathbb{V}$.*

iv. *Unitarity: $1u = u \, \forall u \in \mathbb{V}$.*

v. *Existence of zero: $\exists \ 0 \in \mathbb{V}$ s.t. $u + 0 = u \quad \forall u \in \mathbb{V}$.*

vi. *Negation: For every $u \in \mathbb{V}$, $\exists \ (-u) \in \mathbb{V}$ s.t. $u + (-u) = 0 \in \mathbb{V}$.*

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[1] actually the technically correct terminology is *commutative ring* and not *numbers*.

[2] Linear algebra is the mathematics of vector spaces and their subspaces. Many questions about vector spaces can be reformulated as questions about arrays of numbers (vectors, matrices, tensors)! This is what we will learn in this course.

What's all this fuss about?

Recipe
Should I cook some good math for you?

Definition (recipe): A *recipe* consists of

i. a set I of *ingredients,*

ii. instructions for turning the elements of I into the final dish.

For mathematicians, the *field* and the *set* are like the ingredients of a dish, and the *operations* are like the instructions to cook a dish! The purpose and act of mathematics of vector spaces (cooking using a recipe) are reliant on the field and the set of vectors (ingredients) as well as the operations along with the axioms (instructions).

Examples of vector spaces
What do they look like?

1) Let $\mathbb{V}$ be the set of $n \times 1$ column matrices (vectors), $\mathbb{F}$ be the field of *reals* $\mathbb{R}$, and the laws of vector addition and scalar multiplication are defined as: $\begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \cdot \\ \cdot \\ x_n + y_n \end{pmatrix}$ and $c \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \cdot \\ \cdot \\ cx_n \end{pmatrix}$.

HW: Verify that the above indeed constitutes a vector space!

2) Let $\mathbb{V}$ be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$, let the field of scalars be $\mathbb{R}$, and let the operations be usually defined.
HW: Verify that the above indeed constitutes a vector space!

3) Let $\mathbb{V}$ be the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the equation $f'' = -f$. Can you think of any function that satisfies this property? Cosine, Sine? Let the field of scalars be $\mathbb{R}$. The operations are defined in the usual manner. *Hint: Suppose* $f_1, f_2 \in \mathbb{V}, \quad c \in \mathbb{R};$ *then* $(f_1 + f_2)'' = f_1'' + f_2'' = -f_1 - f_2 = -(f_1 + f_2)$*; and* $(cf_1)'' = cf_1'' = c(-f_1) = -(cf_1)$. *Are these results consistent with the definition of the vector space? Also check whether all axioms are compliant?*

4) Transactions on an accounting system (bank). Think about this for now. We will return to this example soon!

Linear independence of vectors

Do they all look alike? Do they upon each rely?

Definition (<u>Linearly dependent vectors</u>):
Let $\mathscr{V}$ be a vector space and $\mathscr{X} \subset \mathscr{V}$ be a non-empty subset. Then $\mathscr{X}$ is ***<u>linearly dependent</u>*** if there are distinct vectors $v_1, v_2, \ldots, v_k \in \mathscr{X},$ and scalars $c_1, c_2, \ldots, c_k$ (not all of them zero) , s.t. $c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0.$

This is equivalent to saying that *at least one of the vectors* v_i *can be expressed as a linear combination of the others.*

$$v_i = \sum_{j \neq i} -\left(\frac{c_j}{c_i}\right) v_j$$

Definition (<u>Linearly independent vectors</u>):
A subset which is not linearly dependent is said to be *linearly independent.* Thus a set of distinct vectors $\{v_1, v_2, \ldots, v_k\}$ is linearly independent if and only if an equation of the form $c_1v_1 + c_2v_2 + \ldots + c_kv_k = 0$ always implies that $c_1 = c_2 = \ldots = c_k = 0.$