DATE 4/3/2025

Lecture (12):

In the previous lecture we saw a general prescription for a ~~matrix~~ (linear) transformation induced by A.

Infact, and conversely, given a linear transformation (in terms of a differential operator $f'(\cdot) + f''(\cdot)$), we found a way to write down it's matrix representation.

Today, we will discuss a few more matrix (linear) transformations that find wide applications in engineering.

All are examples of linear transformation

(1) Projection operator/transf^n

(2) Shear transformation

(3) Reflection transformation

1) Projection :-

Lets say $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \bar{u} \in \mathbb{R}^3$

We want to obtain the "shadow" of $\bar{u}$ on $\mathbb{R}^2$ (i.e the x-y plane), what should the "matrix ~~multi~~ multiplier" of $\bar{u}$ look like.

It should not be difficult to guess that

$$A\bar{u} = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix} \equiv \bar{u}_{2D}$$

is what suffices.

So the matrix representation of the linear transformation ($O_{g_{proj}}$) is

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix}} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} \end{pmatrix}$$

DATE

We could have gone in the reverse direction.

Consider $T : \mathbb{R}^3 \to \mathbb{R}^2$,

$\hat{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$; $\hat{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$; $\hat{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ are the std. canonical basis vectors in $\mathbb{R}^3$.

We want.

$$T(\hat{e}_1) = T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

a vector on the 2D x-y plane

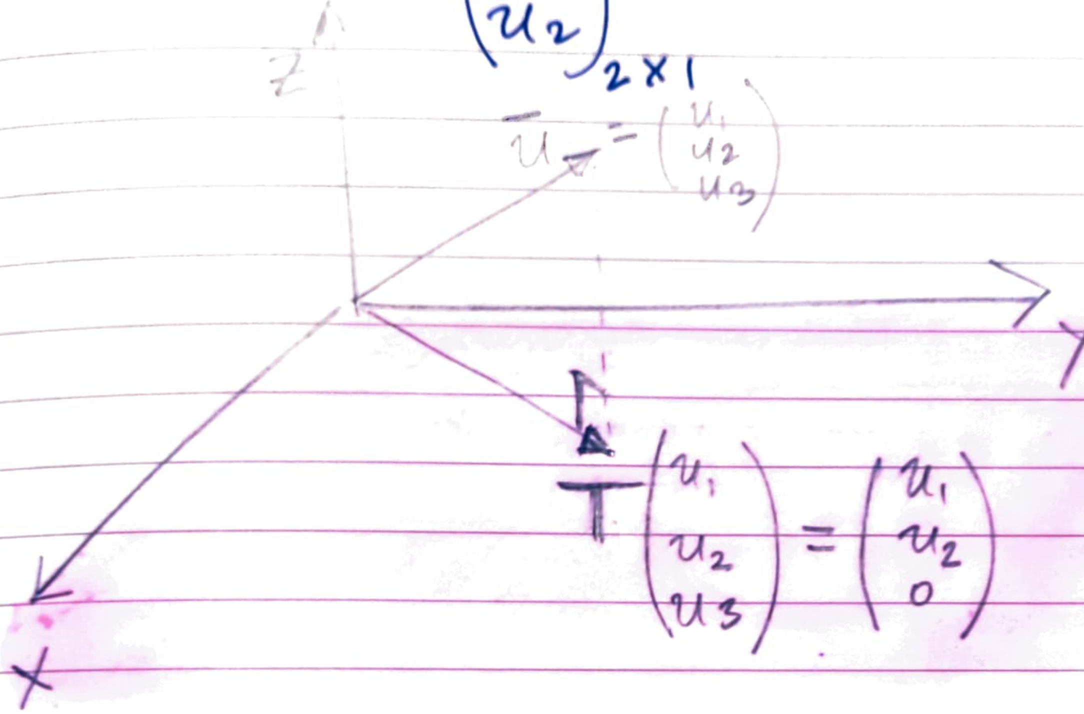
$$T(\hat{e}_2) = T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$$

$$T(\hat{e}_3) = T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$A = \begin{pmatrix} | & | & | \\ T(\hat{e}_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A\bar{u} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{2\times 3} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{3\times 1}$$

$$= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{2\times 1}$$

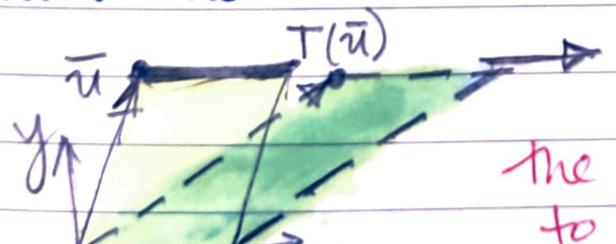


$$T\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ 0 \end{pmatrix}$$

Fig (1) :- orthogonal (OG) proj^n of $\bar{u}$ on the x-y plane.

(2) Shear transformation

What is a shear ?



Effect of shear:

The vector $\bar{u}$ is transformed to $T(\bar{u})$. Clearly, the y-component of the vector has not changed but the x-component has undergone a stretch

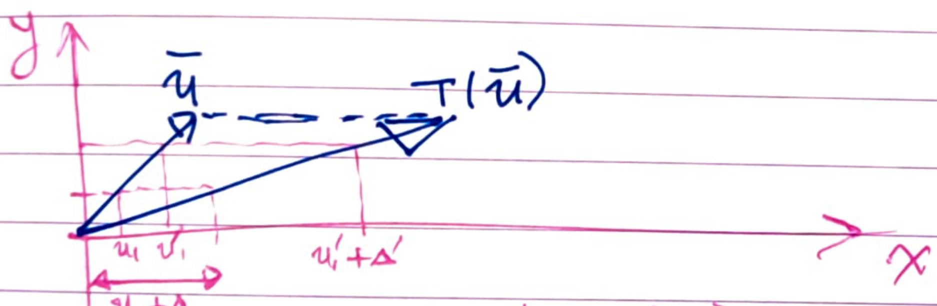
DATE

So what should I expect to be the entries of the matrix multiplier of $\bar{u}$?

$$\begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 + K u_2 \\ u_2 \end{pmatrix}$$

$K = 0$: Identity matrix transform^n (OR no change!)

Why Should the change be $K u_2$??



clearly $\Delta' > \Delta$ & infact it's more in linear proportion to the ht. (or u_2)

here, $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$T(\hat{e}_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + K(0) \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$T(\hat{e}_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} K(1) \\ 1 \end{pmatrix} = \begin{pmatrix} K \\ 1 \end{pmatrix}$$

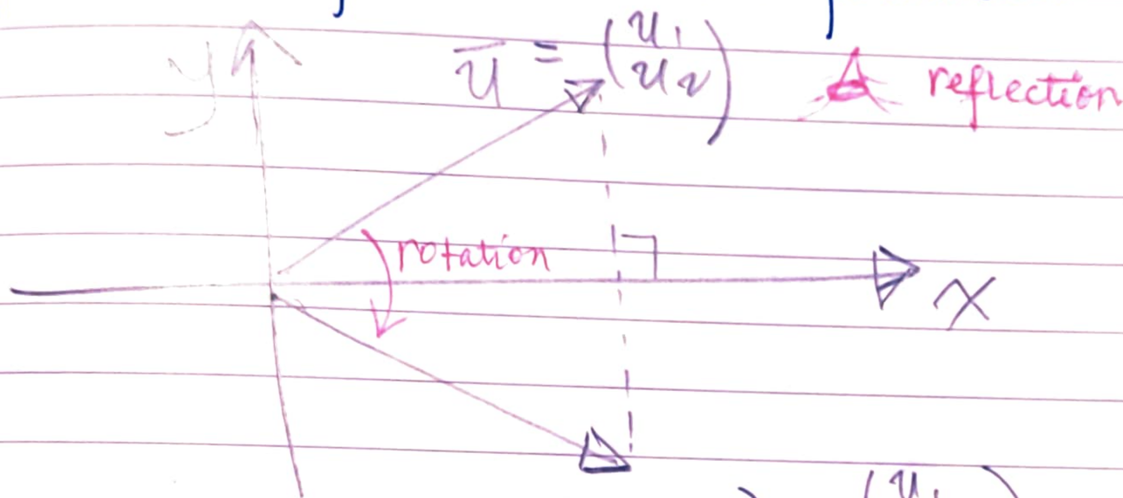
$\therefore \quad A = \begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}$

DATE

(3) Reflection (about the x-axis)

Is reflection really different from rotation?

Let's begin w/ the visual picture of this transformation



Want: $T(\vec{u}) = \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$

So, $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$\left.\begin{aligned} T\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ T\begin{pmatrix} 0 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ -1 \end{pmatrix} \end{aligned}\right\} \to A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$T(\bar{u}) = A\bar{u} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\bar{u} = \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$ Exactly what I wanted!