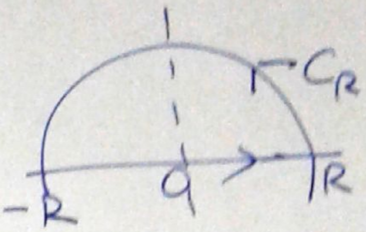
18/4/19

Lecture (18)

Application / Example of Th^m (17.2) $\left(\int_{C_R} f(z)\,dz \xrightarrow{R\to\infty} 0 \right)$

Q) Evaluate :- $I = \int_{-\infty}^{\infty} \frac{dx}{x^4+1}$



$$f(z) = \frac{N(z)}{D(z)} = \frac{1}{z^4+1}$$

$$\oint_C f(z)\,dz = 2\pi i \sum_{\substack{z_j \in s.p. \\ w/i\ C}} \text{Res}(f(z); z_j) = \int_{-\infty}^{\infty} f(x)\,dx + \lim_{R\to\infty} \underbrace{\int_{C_R} f(z)\,dz}_{0}$$

(b/c Th^m(17.2) applies) — ①

Let us find the roots of z^4+1 to find s.p. of $f(z)$.

$z^4+1=0$

$\Rightarrow z^4 = -1$

$\Rightarrow z^2 = \pm i$

i.e. $z^4 = -1$

$\Rightarrow R^4(\cos 4\theta + i\sin 4\theta) = -1$ — De Moivre's th^m

$\Rightarrow R=1;\ \theta = \pi/4$

$R=1;\ \theta = \pi/4 + \pi/2 = 3\pi/4$

These are 2 isolated s.p. of $f(z)$ in UHP. The remaining s.p. are in L.H.P so we don't care abt them

$z_1 = e^{i\pi/4}$

$z_2 = e^{i3\pi/4}$

$$\operatorname{Res}(f(z); z_1) = \left.\frac{1}{4z^3}\right|_{z_1 = e^{i\pi/4}} = \frac{1}{4e^{i3\pi/4}}$$

Here we have used

$$\operatorname{Res}(f(z); z_1) = \cancel{\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}}$$

$$= \boxed{\frac{N(z_1)}{D'(z_1)}}$$ b/c all poles are simple

$$= \left(\frac{1}{4z^3}\right)_{z_1 = e^{i\pi/4}}$$

$$= \frac{1}{4} e^{-i3\pi/4}.$$

Likewise $\operatorname{Res}(f(z); z_2) = \left.\frac{1}{4z^3}\right|_{z_2 = e^{i3\pi/4}}$

$$= \frac{1}{4} e^{-i9\pi/4}$$

from eq(1) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{dx}{x^4+1}$

$$= 2\pi i \left(\frac{1}{4e^{i3\pi/4}} + \frac{1}{4e^{i9\pi/4}} \right)$$

$$= \frac{2\pi i}{4} \left(e^{-i(\pi/2 + \pi/4)} + e^{-i\pi/4} \right)$$

$$= \frac{\pi i}{2} \Big(\underbrace{\cos(\pi/2 + \pi/4)}_{-\sin \pi/4} - i \overbrace{\sin(\pi/2 + \pi/4)}^{\cos \pi/4}$$

$$+ \cos \pi/4 - i \sin \pi/4 \Big)$$

$$= \frac{i\pi}{2} \left\{ \cos \pi/4 - \sin \pi/4 - i\left(\cos\frac{\pi}{4} + \sin \pi/4\right) \right\}$$

pg (2)

$$= \frac{i\pi}{2}\left\{ \cos\left(\cancel{\pi/2} - \pi/4\right) - \sin \pi/4 \right.$$

$$\left. - i\left(\cos \pi/4 + \sin \pi/4\right)\right\}$$

$$= \frac{\pi}{2}\left(\cos \pi/4 + \sin \pi/4\right)$$

$$= \frac{\pi}{\sqrt{2}}.$$

#.

§ (18.1) Argument Principle.

$$f(z) = |f(z)| e^{i \arg(f(z))}$$

Notation :- $\left(\arg(f(z))\right)_C$:= change in $\arg(f(z))$ over C

Let $f(z)$ be a meromorphic f^n defined inside & on a Jordan curve C w/ no zeros/poles on C.

then $\boxed{I = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = N - P = \frac{1}{2\pi}\left\{\arg(f(z))\right\}_C}$

Where N = no. of zeros of $f(z)$ inside C

P = no. of poles of $f(z)$ inside C

§(18.2)

Rouché's Th^m

Let $f(z)$ & $g(z)$ be analytic on & inside a Jordan contour C,

If $|f(z)| > |g(z)|$ on C;

then

$f(z)$ and $(f(z)+g(z))$ have the same no. of zeros inside C.

Application (Nxt. Class)

Rouché's Th^m is useful for proving the Fundamental th^m of Algebra i.e.

$P_n(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

has n roots counting multiplicity.

#.

Next Lecture :- More examples on the (above) concepts of this lecture!