Solving systems of linear eqns. (iterative methods) (1)

Q1) $x_1 - x_2 + x_3 + x_4 = 2$ ——— (1)

$x_1 + x_2 + x_3 - x_4 = 3$ ——— (2)

$x_1 + 3x_2 + x_3 - 3x_4 = 1$ ——— (3)

Does this set of eqns. have a soln. ?

Lets perform the following operations to eliminate x_1

eq(2) : eqn(2) − eqn(1) and eqn(3): eqn(3) − eqn(1)

$x_1 - x_2 + x_3 + x_4 = 2$ ——— (1)

$2x_2 - 2x_4 = 1 \Big) \frac{1}{2} \Rightarrow x_2 - x_4 = \frac{1}{2}$ ——— (2)

$4x_2 - 4x_4 = -1$ ——— (3)

Next we will attempt to eliminate x_2 from eq (3);

(3) : (3) − 4(2) to get

$$\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 2 \\ x_2 \qquad - x_4 &= \tfrac{1}{2} \\ 0 &= -3 \end{aligned}$$

oops??..

the 3rd eq. above is obviously false; so the system of eqns. has "no" solns. i.e. it is inconsistent.

Of course this was expected b/c 3 eqns & 4 unknowns

Q2)

$$x_1 + 4x_2 + 2x_3 = -2 \quad \text{———} \quad (1)$$
$$-2x_1 - 8x_2 + 3x_3 = 32 \quad \text{———} \quad (2)$$
$$x_2 + x_3 = 1 \quad \text{———} \quad (3)$$

What about this sys. of eqns.?

(2): (2) + 2(1) gives.

$$x_1 + 4x_2 + 2x_3 = -2$$
$$7x_3 = 28$$
$$x_2 + x_3 = 1$$

Swap the order of these 2 eqns.

$$x_1 + 4x_2 + 2x_3 = -2 \quad \text{———} \quad (1)$$
$$x_2 + x_3 = 1 \quad \text{———} \quad (2)$$
$$7x_3 = 28 \quad \text{———} \quad (3)$$

(3) : $\frac{1}{7}(3)$

$$x_1 + 4x_2 + 2x_3 = -2 \text{ ——— } (1)$$
$$x_2 + x_3 = 1 \text{ ——— } (2)$$
$$x_3 = 4 \text{ ——— } (3)$$

Now we know $\underline{x_3 = 4}$!

Now perform "<u>Back substitution</u>" to find

$$x_2 = 1 - x_3 = -3$$

and

$$x_1 = -2 - 4x_2 - 2x_3 = 2$$

this linear system of eq. has a unique soln.!

Q3) $x_1 + 3x_2 + 3x_3 + 2x_4 = 1$ ——— (1)
$2x_1 + 6x_2 + 9x_3 + 5x_4 = 5$ ——— (2)
$-x_1 - 3x_2 + 3x_3 = 5$ ——— (3).

(2) : (2) − 2(1) and (3) : (3) + (1)

$x_1 + 3x_2 + 3x_3 + 2x_4 = 1$ ——— (1)
$3x_3 + x_4 = 3$ ——— (2)
$6x_3 + 2x_4 = 6$ ——— (3)

Note x_2 has disappeared from eqs. (2) & (3); so we proceed to the next unknown x_3!

(2) : $\frac{1}{3}$(2) followed by (3) : (3) − 6(2)

$x_1 + 3x_2 + 3x_3 + 2x_4 = 1$ ——— (1)
$x_3 + \frac{1}{3}x_4 = 1$ ——— (2)
$0 = 0$ ——— (3)

Here the 3^{rd} eqn. tells us nothing & can be ignored. (6)

Now we can make the following arbitrary assignments to x_4 and x_2

$$x_4 = c \; ; \quad x_2 = d \; ;$$

to recover the two other unknowns

by using back-substitution $\left\{ \begin{array}{l} x_1 = -2 - c - 3d \; ; \\ x_3 = 1 - \frac{c}{3} \end{array} \right.$

This linear sys. has ∞ly many sol^n's.

What have we learnt?

1) the operational mechanism of Gauss Elimination

11) Echelon form, pivots, etc...