Lecture (6) : part (II) 27/1/19.

In the previous lecture, we found that $\log z$ has 2 branch pts. viz. $z=0$ and $z=\infty$ and $\operatorname{Re} z>0$ is the branch cut. This enabled us to define a cut plane: $\{ \mathbb{C} - \{z=0, z=\infty, \operatorname{Re} z>0\}\}$ where $\log z$ is single valued and continuous.

Here we will show below that $\log z$ is analytic in the cut plane where $\frac{d \log z}{dz}$ is $\frac{1}{z}$.

We will establish this via the Cauchy-Riemann eqs.

$z = x+iy$

$\omega = \log z = u+iv$

$$e^{2u} = e^{2\operatorname{Re}(\omega)} = e^{2\log r} = e^{\log r^2}$$

b/c $u = \log r$ from part (1) of Lecture (6)

$$\Rightarrow \boxed{e^{2u} = r^2 = x^2+y^2}$$

$$e^{\omega} = z = e^{u}e^{iv} = r(\cos v + i \sin v)$$

$z = x+iy$

$$\Rightarrow \boxed{\tan v = \frac{y}{x}}$$

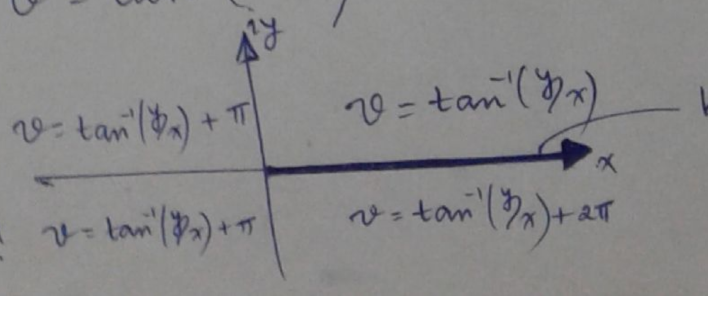
$$\log e^{2u} = \log(x^2+y^2)$$

$$\Rightarrow u = \frac{1}{2}\log(x^2+y^2) \quad \text{— (1)}$$

$x, y = 0$ is branch pt & hence ignored, so this def^n of u suffices.

$$v = \tan^{-1}(y/x) + c_i \quad \text{— (2)}$$

v is continuous in z-plane except $\operatorname{Re} z>0$ where there is a jump of 2π across $\operatorname{Re} z>0$!



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this choice of u and v is appropriate to make $\log z$ continuous & single valued in the cut plane : $\mathbb{C} - \{ z=0, z=\infty, \operatorname{Re} z > 0 \}$.

Now

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{x}{x^2+y^2} \, , \quad \dfrac{\partial u}{\partial y} = \dfrac{y}{x^2+y^2} \\[2ex] \dfrac{\partial v}{\partial x} = \dfrac{-y}{x^2+y^2} \, ; \quad \dfrac{\partial v}{\partial y} = \dfrac{x}{x^2+y^2} \end{cases}$$

b/c

$$\frac{\partial}{\partial x} \tan^{-1}(y/x) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

& likewise for $\dfrac{\partial v}{\partial y}$.

$$\Rightarrow \left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ &\text{and} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}\right\} \text{CR conditions.}$$

$\Rightarrow \log z$ is analytic in the cut plane.

Finally,

$$\begin{aligned} \frac{d}{dz} \log z &= \frac{d}{dx}(u+iv) \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right) \\ &= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{z} \end{aligned}$$

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