Types of probability distributions & their applications.

Discrete prob. D^n.

1) Bernoulli
2) Binomial
3) Geometric
   - Type 1
   - Type 2
4) Poisson
5) Discrete Uniform D^n.

Continuous prob. D^n.

1) Normal / Gaussian
2) Exponential
3) Pareto
4) Beta
5) Continuous Uniform D^n
6) χ^2
7) t - D^n
8) F - D^n.

Bernoulli D^n

Heads / Tail

Success / failure

Binary outcomes

$$X \sim \text{Bernoulli}(p)$$

$$f_X(x) \equiv P(X = x) = \begin{cases} p \;; & x = 1 \\ 1-p \;; & x = 0 \end{cases}$$

$$E(X) = \sum_{x \in \Omega} x \, P(X=x)$$

$$E(X^2) = \sum_{x \in \Omega} x^2 . P(X=x)$$

$$E(X) = 1 \times p + 0(1-p) = p$$

$$Var(X) = E(X^2) - (E(X))^2 = \{p(1)^2 + (1-p)(0)^2\} - p^2 = p(1-p).$$

Binomial D^n.

$$Y \sim Bin(n, p) \qquad {}^nC_r = \frac{n!}{r!(n-r)!}$$

$Y = y$ successes in n independent trials w/ success probability $= p$.

$$f_Y(y) = P(Y=y) = {}^nC_y\, p^y (1-p)^{n-y} \quad ; \quad y = 0, 1, 2, \ldots, n$$

$$E(Y) = \sum_{y=0}^{n} y * P(Y=y) = 0 * P(Y=0) + 1 * P(Y=1) + 2 * P(Y=2) + \cdots + n * P(Y=n)$$

$$= 1 * {}^nC_1\, p (1-p)^{n-1} + 2 * {}^nC_2\, p^2 (1-p)^{n-2} + \cdots + n * {}^nC_n\, p^n (1-p)^0$$

$$= np(1-p)^{n-1} + 2\frac{n(n-1)}{2} p^2 (1-p)^{n-2} + \cdots + np^n$$

$$= \cdots = np.$$

$$Var(Y) = np(1-p)$$

eg. Binomial D^n.

Q) A coin is tossed 3 times. What is the probability that you observe 2 Heads as outcomes of these 3 tosses? Assume a fair coin!

$p_H = 1/2 = p_T$

$n = 3$

Ans) $P(Y=2) = {}^3C_2 \, p_H^2 (1-p_H) = \frac{3!}{2!\,1!} \left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)$

$= \frac{3 \times 2}{2} \; \frac{1}{8}$

$= \frac{3}{8}.$

* If $Y_1 \sim \text{Bin}(n, p)$

$Y_2 \sim \text{Bin}(m, p)$

$Y_1 + Y_2 \sim \text{Bin}(n+m, p).$

Ans) i) $P(X>3) = 1 - P(X \le 3) = 1 - \sum_{k=0}^{3} \frac{e^{-2}\lambda^k}{k!}$

$= 1 - e^{-2}\left\{1 + 2 + \frac{4}{2} + \frac{8}{6}\right\}$

≈ 0.143.

Note. $\sum_{k=0}^{3} \frac{\lambda^k}{k!} = \left\{\frac{2^0}{1} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!}\right\}$

$= \left\{1 + 2 + \frac{4}{2} + \frac{8}{6}\right\}$

ii) $P(X=0) = \frac{e^{-2}\lambda^0}{0!} = e^{-2} \approx 0.135$

Geometric D^n

Type (1) :- X = no. of trials until 1^{st} Success

$X \sim geom_1(p)$

$$P(X=x) = \begin{cases} (1-p)^{x-1} p & ; x = 1, 2, \dots \\ 0 & ; \text{otherwise} \end{cases}$$

$E(X) = \frac{1}{p}$; $Var(X) = \frac{(1-p)}{p^2}$

Type (0)

Y = no. of failures before 1^{st} success

$$P(Y=y) = \begin{cases} (1-p)^y p & ; y = 0, 1, \dots \\ 0 & ; \text{otherwise} \end{cases}$$

$Y \sim geom_0(p)$ $E(Y) = \frac{1-p}{p}$

$Var(Y) = \frac{1-p}{p^2}$

Poisson D^n $\equiv$ No. of arrivals in a given time interval.

$X \sim$ poisson (λ) ; $\lambda \in \mathbb{R}$ is rate of arrival.

$$f_X(x) = P(X=x) = \left(\frac{\lambda^x e^{-\lambda}}{x!} \right)$$

$$\left.\begin{aligned} E(X) &= \lambda \\ Var(X) &= \lambda \end{aligned}\right\} \text{Both are same!}$$

Q) The no. of telephone calls arriving at a switchboard during any 10 min. period is known to be a Poisson random variable (RV) X w/ $\lambda = 2$ (rate)

(i) Find the probability that more than 3 calls will arrive during any 10 min. period.

(ii) Find the probability that no calls will arrive during any 10-min. period.

eg. Geometric D^n.

Q) If your probability of success of meeting a congress voter is 0.2; what is the probability you will meet a congress voter on your 3^{rd} meeting?

Ans) $p = 0.2$

X = no. of trials until 1^{st} success (including the successful meeting)

$X \sim geom(p)$

$$P(X=3) = (1-p)^{3-1} p = (0.8)^2 (0.2) = 0.128.$$

Another eg. of a disc. RV.

Q) A binary source generator generates digits 0 and 1 randomly w/ probabilities 0.4 and 0.6 respectively.

i) What is the probability that two 1s and three 0s will occur in a 5-digit sequence?

ii) What is the probability that atleast three 1s will appear in a 5-digit seq.?