4/10/2018

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(5) 8.7

Alternating Series Test (Leibniz's theorem).

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots \quad \text{Converges}$$

if "all" of the following hold true :-

(I) $u_n > 0 \quad \forall n$

(II) $u_n \geq u_{n+1} \quad \forall n \geq N \in I^+$

(III) $u_n \to 0$

Proof:- Let $n = 2m$

$$S_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots + (u_{2m-1} - u_{2m}) \quad \text{— ①}$$

$$= u_1 - (u_2 - u_3) - (u_4 - u_5) - \cdots - (u_{2m-2} - u_{2m-1}) - u_{2m} \quad \text{— ②}$$

Let $N = 1$

$\because u_n \geq u_{n+1} \quad \forall n \geq N$ $\Big\} \Rightarrow S_{2m}$ is sum of m non-negative terms (from ① above)

$S_{2m+2} = S_{2m} + (u_{2m+1} - u_{2m+2})$ is the sum of $(m+1)$ non-negative terms

& $S_{2m+2} \geq S_{2m}$ i.e. $\{S_{2m}\}$ is a non-decreasing sequence. — (A)

from ② above,

$S_{2m} \leq u_1$ $\forall m$ b/c $S_{2m} + \underbrace{(u_2 - u_3)}_{\geq 0} + \underbrace{(u_4 - u_5)}_{\geq 0} + \cdots + \underbrace{u_{2m}}_{\geq 0} = u_1 < \infty$

i.e. $\{S_{2m}\}$ is bdd from above — (B)

Using Monotone Seq. theorem, (A) & (B) $\Rightarrow$

$$\lim_{m \to \infty} S_{2m} = L < \infty \quad \text{— ③}$$

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If $n = 2m+1$;

$$s_{2m+1} = s_{2m} + u_{2m+1}$$

$\because u_n \to 0 \Rightarrow \lim_{m\to\infty} u_{2m+1} = 0$

and,

$$\lim_{m\to\infty} s_{2m+1} = \lim_{m\to\infty} s_{2m} + 0$$
$$= L < \infty \quad \text{——(4)}$$

(3) & (4) $\Rightarrow \lim_{n\to\infty} s_n = L < \infty$

i.e. the alternating series $u_1 - u_2 + u_3 - u_4 + \cdots$ converges.

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Example :-

(1) Test the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$

We have $u_n = \frac{1}{n} > 0$ & $\lim_{n\to\infty} \frac{1}{n} = 0$

$u_{n+1} = \frac{1}{n+1} < \frac{1}{n} = u_n \quad \forall n \geq 1$

$\therefore$ The alternating series converges

Note :- that an alternating series converges while the sum of the absolute values diverges (cmp. above series w/ harmonic series)

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Example (2) :-

Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$ for convergence.

We have $u_n = \frac{n}{n^2+1} > 0$

$$\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^2} = 0$$

to check for $b_{n+1} \le b_n$; we test ~~the~~ increasing/decreasing behavior of the f^n $f(x) = \frac{x}{x^2+1}$ w/ $f(n) = b_n$

$$f'(x) = \cdots = \frac{1-x^2}{(x^2+1)^2} < 0 \quad \forall x > 1$$

$\Rightarrow b_{n+1} \le b_n \quad \forall n > 1$

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ converges.

** Before trying to apply the alternating series test always try/check

if $\lim_{n\to\infty} u_n = 0$ or NOT

b/c if it is not then the corresponding series diverges by ~~the~~ n^{th} term test.

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Thm (Alternating Series Estimation theorem).

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the 3 conditions of thm (Leibniz); then for $n \geq N$,

$$S_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n \approx L$$ (Sum of the ∞ Series)

w/ error whose absolute value is $\leq u_{n+1}$ (the numerical value of the 1st unused term)

Also,
$\text{sign}(L - S_n) = \text{sign}(u_{n+1})$.

Example

(1) $$\sum(-1)^n \frac{1}{2^n} = \frac{1}{1-(-\frac{1}{2})} = 2/3 = L$$

Let's truncate after 8th term

$$\underbrace{1 - 1/2 + 1/4 - 1/8 + 1/16 - 1/32 + 1/64 - 1/128}_{0.6640625} + \frac{1}{256}$$

$L - S_8 = 2/3 - 0.6640625 = 0.002604166$

is +ve & $< \frac{1}{256} = 0.00390625$

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§(8.7) Contd...

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Absolute Convergence

Def^n A series $\sum a_n$ converges absolutely if the corresponding series of absolute values $\sum |a_n|$ converges.

eg the geom series $1 - 1/2 + 1/4 - \cdots$

Converges absolutely b/c

$1 + 1/2 + 1/4 + \cdots$ Converges.

Def^n A series that does not converge absolutely converges conditionally.

$\sum (-1)^n \frac{1}{n}$ converges conditionally

b/c $\sum \left| (-1)^n \frac{1}{n} \right| = \sum \frac{1}{n}$ diverges.

Th^m (Absolute Convergence test).

If $\sum_{n=1}^{\infty} |a_n|$ converges; then $\sum_{n=1}^{\infty} a_n$ converges

Why?? for each n,

$$-|a_n| \le a_n \le |a_n|$$

$$\Rightarrow 0 \le a_n + |a_n| \le 2|a_n| \quad \text{— ①}$$

Now if $\sum |a_n|$ converges $\Rightarrow \sum 2|a_n|$ converge

& by applying the direct comparison test by noting ① we have.

$\sum (a_n + |a_n|)$ converges.

Further, b/c $a_n = (a_n + |a_n|) - |a_n|$

we have $\sum a_n = \overset{\text{converges}}{\sum a_n + |a_n|} - \overset{\text{Converges}}{\sum |a_n|} \Rightarrow$

$\sum a_n$ converges !

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eg ① $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$

is convergent b/c it is "absolutely convergent" since $\sum \frac{1}{n^2}$ converges by p-series test.

eg ② $\sum_{n=1}^{\infty} \frac{\sin n}{n^2} = \frac{\sin 1}{1} + \frac{\sin 2}{4} + \cdots$

Converges b/c $\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right|$ conv. by comparison with the convergent $\sum \frac{1}{n^2}$ series & the fact $|\sin n| \le 1 \ \forall n$.

eg ③ Alternating p series $(p > 0)$

$\frac{1}{n^p}$ is a decreasing sequence w/ limit 0.

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots, \ p > 0$

converges

Converges absolutely for $p > 1$

Converges conditionally for $0 < p \le 1$.

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th^m (Rearrangement theorem for absolutely convergent series:).

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, & $b_1, b_2, \ldots, b_n, \ldots$ is any re-arrangement of the sequence $\{a_n\}$ then $\sum b_n$ converges absolutely &

$$\sum_{n\geq 1} b_n = \sum_{n\geq 1} a_n$$

eg: the series $\sum_{n\geq 1} a_n = 1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} - \frac{1}{36} - \frac{1}{64} - \frac{1}{100} - \frac{1}{144} + \cdots$

(after k terms of one sign, take k+1 terms of the other sign).

Rearrange as:

$$\sum_{n\geq 1} b_n = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots + (-1)^{n-1}\frac{1}{n^2} + \cdots$$

converges absolutely (from earlier result)

$\Rightarrow \sum_{n\geq 1} a_n$ converges.

eg (Abs & Cond. conv.).

Analyse the conv./div. of $\displaystyle\sum_{n=2}^{\infty} \frac{(\sin n) + 1/2}{n(\ln n)^2}$

Soln:- $\displaystyle\left|\frac{\sin n + 1/2}{n(\ln n)^2}\right| \le \frac{|\sin n| + 1/2}{n(\ln n)^2} \le \frac{1 + 1/2}{n(\ln n)^2}$

If we can show $\displaystyle\sum_{n\ge 2} \frac{1}{n(\ln n)^2}$ Conv. then by direct comparison test $\displaystyle\sum_{n\ge 2} \left|\frac{\sin n + 1/2}{n(\ln n)^2}\right|$ conv and then in turn by absolute Convergence test $\displaystyle\sum_{n\ge 2} \frac{(\sin n) + 1/2}{n(\ln n)^2}$ conv.!

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So what about $\cancel{\sum_{n\ge 2} (\sin n) + 1/2}$ $\displaystyle\sum_{n\ge 2} \frac{1}{n(\ln n)^2}$??

Test conv/div. of $\displaystyle\sum_{n\ge 2} \frac{1}{n(\ln n)^k}$; $k > 1$

Let $f(x) = \frac{1}{x(\ln x)^k}$ on $[2, \infty)$; $f(x) > 0$ & $\because f'(x)$

$= x^{-2}(\ln x)^{-k} - kx^{-1}(\ln x)^{-k-1}\frac{1}{x} = -x^{-2}(\ln x)^{-k-1}(\ln x + k)$

≤ 0 when $\ln x > -k$

$\Rightarrow f(x)$ is monotone decreasing when $\ln x > -k$ i.e. $f(x)$ is eventually monotonically decreasing

no.

$\displaystyle\int_2^{\infty} f\,dx \;\underset{dy = \frac{1}{x}dx}{\overset{y = \ln x}{=}}\; \int_{\ln 2}^{\infty} 1/y^k\,dy = \begin{cases} y^{-k}/(-k+1)\Big|_{\ln 2}^{\infty} & k \neq 1 \\ \ln(+\infty) - \ln(\ln 2) & k = 1 \end{cases} \Rightarrow \int_2^{\infty} f\,dx < \infty$ iff $k > 1$ #.