Engineering Mathematics in Action: FM 103 — 9th October, 2023

Mid-semester examination — Total time: 3 hours

Instructions: You must **not** be in possession of any cheat sheet, notes, or electronic devices like laptops or calculators inside the examination hall. Please answer all five questions. Please begin your answer to a given question on a new page. Please show all steps leading to your final answer to receive any credit for your solution. Merely stating the final answer may not fetch you any credit. Maximum point allotted to each question is mentioned in the square bracket on the right margin. Maximum score of this examination is **30**.

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1. Geometric interpretation of vectors in vector spaces

(i) The scalar product of any two vectors is defined as follows: $\langle \mathbf{v}, \mathbf{w} \rangle = |\mathbf{v}||\mathbf{w}|\cos\theta$, where θ is the angle between the two vectors in the counter-clockwise sense. Consider two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ such that $\mathbf{v_1} \perp \mathbf{v_2}$. Compute $\langle \mathbf{v_1}, \mathbf{v_2} \rangle$. **[1]**

(ii) Consider a system of linear equations $A\mathbf{x} = \mathbf{b}$. The matrix $A = \begin{pmatrix} 1 & 3 & 9 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & 2 & 6 \end{pmatrix}$ and the vector $\mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \end{pmatrix}$. Guess the solution $\mathbf{x}$ by *inspecting* the matrix A and the vector $\mathbf{b}$. Explain your answer in detail. **[5]**

NOTE: You are **not** allowed to use any matrix transformations like row transformations, you are **not** allowed to bring either A or $\tilde{A}$ to either the row-echelon form or reduced row echelon form, you are **not** allowed to use Gauss elimination or Gauss-Jordan elimination or LU factorization methods to answer this question. You are **not** allowed to solve the system of equations algebraically or by method of substitution. You are **not** allowed to use Cramer's method to answer this question.

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2. Linear Transformation

(i) Find the matrix representation B of the linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$ defined as follows: $T\big(f(x)\big) = f'(x) + f''(x)$. Use the standard basis $\mathfrak{B} = \{1, x, x^2\}$. **[4]**

(ii) Consider the polynomial $q(x) = -x + 3$. Compute $T(q)$ using the matrix B. **[2]**

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3. Null space of a matrix

Consider the matrix $A = \begin{pmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{pmatrix}$. Find the *basis* of the vector space $\mathscr{V} = null(A)$. Then find the dimension of $\mathscr{V}$. **[5+1]**

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4. Column space of a matrix

Consider the matrix $A = \begin{pmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{pmatrix}$. Find the *basis* of the vector space $\mathscr{W} = col(A)$. Then find the dimension of $\mathscr{W}$. **[5+1]**

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5. LU factorization

Consider the system of equations given below.

$$x_1 + 4x_2 + 2x_3 = -2$$
$$-2x_1 - 8x_2 + 3x_3 = 32$$
$$x_2 + x_3 = 1$$

Write this system in the form $A\mathbf{x} = \mathbf{b}$. Find the LU factorization of A. Then use this factorization to find the solution $\mathbf{x}$. **[1+3+2]**

========================== **END OF QUESTION PAPER** ==========================