Tutorial Worksheet-2 (WL2.2, WL3.1)
Define vector spaces of m×n matrices and its practical applications, introduction to system of linear equations, Row-Reduced Echelon form, rank of a matrix, linear transformation

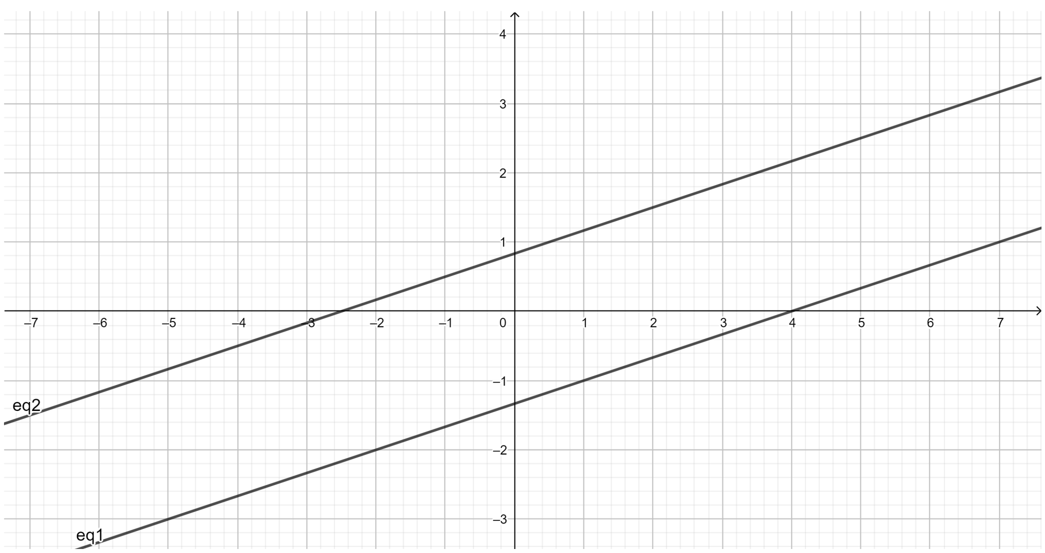
Name and section: ______________________________

Instructor's name: ______________________________

1. Check the consistency of the following system of equations graphically:

$$\begin{aligned} x - 3y &= 4 \\ -2x + 6y &= 5 \end{aligned}$$

**Solution:** Construct the two equation on same graph paper:




source: geogebra.org

Since the two equations never meet (as it seems parallel to each other), therefore the solution is inconsistent, that is, the system has no solution.

2. Prove that the set of Matrices of order $2 \times 3$ denoted as $\mathbb{M}_{2\times3}(\mathbb{R})$ forms a vector space over $\mathbb{R}$ under usual addition and scalar multiplication of matrices.

**Solution:**

Let $A, B$, and $C \in \mathbb{M}_{2\times3}(\mathbb{R})$ such that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix},\ B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix},\ and\ C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

i. **Closure of Addition:-** If $A, B \in \mathbb{M}_{2\times3}(\mathbb{R})$ then $A + B \in \mathbb{M}_{2\times3}(\mathbb{R})$
Let

$$A+B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} \in \mathbb{M}_{2\times3}(\mathbb{R})$$

Hence, closure of addition property holds.

ii. **Closure of Scalar Multiplication:-** If $A \in \mathbb{M}_{2\times3}(\mathbb{R})$ then $c \cdot A \in \mathbb{M}_{2\times3}(\mathbb{R})$.

$$c \cdot A = \begin{bmatrix} c\cdot a_{11} & c\cdot a_{12} & c\cdot a_{13} \\ c\cdot a_{21} & c\cdot a_{22} & c\cdot a_{23} \end{bmatrix} \in \mathbb{M}_{2\times3}(\mathbb{R})$$

Hence, closure of scalar multiplication property holds.

iii. **Commutativity of Addition:-** For all $A, B \in \mathbb{M}_{2\times3}(\mathbb{R})$, $A + B = B + A$.

$$A+B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = B + A$$

Hence, commutativity of addition property holds.

iv. **Associativity of Addition:-** For all $A, B, C \in \mathbb{M}_{2\times3}(\mathbb{R})$ , $(A + B) + C = A + (B + C)$.

$$(A+B)+C = \left( \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \right) + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} \\ b_{21}+c_{21} & b_{22}+c_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \left( \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} \right)$$

$= A + (B + C)$ Hence, associativity of addition property holds.

v. **Additive Identity:-** For every $A \in \mathbb{M}_{2\times3}(\mathbb{R})$ there exist an element called the zero element and denoted $0 \in \mathbb{M}_{2\times3}(\mathbb{R})$ such that $A + 0 = A$.

Let us take

$$A+0 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

Hence, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the additive identity of $\mathbb{M}_{2\times3}(\mathbb{R})$.

vi. **Additive Inverse:-** For each element $A \in \mathbb{M}_{2\times3}(\mathbb{R})$ there is an element $D \in \mathbb{M}_{2\times3}(\mathbb{R})$ such that $A + D = 0$.
Let us take

$$A + D = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \end{bmatrix}$$

is the inverse element for $A \in \mathbb{M}_{2\times3}(\mathbb{R})$.

vii. **Scalar Identity** For each $A \in \mathbb{M}_{2\times3}(\mathbb{R}), 1 \cdot A = A$.

$$1 \cdot A = 1 \cdot \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 1 \cdot a_{11} & 1 \cdot a_{12} & 1 \cdot a_{13} \\ 1 \cdot a_{21} & 1 \cdot a_{22} & 1 \cdot a_{23} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = A$$

viii. **Scalar Associativity:-** For all $A \in \mathbb{M}_{2\times3}(\mathbb{R})$ and $a, b \in \mathbb{R}$, $(ab)A = a(bA)$.

$$(ab) \cdot A = (ab) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} ab \cdot a_{11} & ab \cdot a_{12} & ab \cdot a_{13} \\ ab \cdot a_{21} & ab \cdot a_{22} & ab \cdot a_{23} \end{bmatrix} = a \begin{bmatrix} b \cdot a_{11} & b \cdot a_{12} & b \cdot a_{13} \\ b \cdot a_{21} & b \cdot a_{22} & b \cdot a_{23} \end{bmatrix}$$

$$= a\left( b \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \right) = a(b \cdot A)$$

ix. **Scalar Distribution:-** For all $A, B \in \mathbb{M}_{2\times3}(\mathbb{R})$ and $a \in \mathbb{R}$, $a \cdot (A + B) = a \cdot A + a \cdot B$.

$$a \cdot (A + B) = a\left( \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} \right)$$

$$= \left( \begin{bmatrix} a \cdot a_{11} & a \cdot a_{12} & a \cdot a_{13} \\ a \cdot a_{21} & a \cdot a_{22} & a \cdot a_{23} \end{bmatrix} + \begin{bmatrix} a \cdot b_{11} & a \cdot b_{12} & a \cdot b_{13} \\ a \cdot b_{21} & a \cdot b_{22} & a \cdot b_{23} \end{bmatrix} \right)$$

$$= a \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + a \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = a \cdot A + a \cdot B.$$

x. **Vector Distribution:-** For all $A \in \mathbb{M}_{2\times3}(\mathbb{R})$ and $a, b \in \mathbb{R}$, $(a + b) \cdot A = a \cdot A + b \cdot A$.
Take,

$$(a + b) \cdot A = a \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} + b \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = a \cdot A + b \cdot A.$$

3. For which values of the constant $c$ is $\begin{bmatrix} 1 \\ c \\ c^2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$.

**Solution:** suppose $\begin{bmatrix}1\\c\\c^2\end{bmatrix}$ is linear combination of $\begin{bmatrix}1\\0\\4\end{bmatrix}$, and $\begin{bmatrix}1\\3\\9\end{bmatrix}$ then $\exists\, c_1, c_2$ such that

$$c_1\begin{bmatrix}1\\0\\4\end{bmatrix}+c_2\begin{bmatrix}1\\3\\9\end{bmatrix}=\begin{bmatrix}1\\c\\c^2\end{bmatrix}$$

The augmented matrix will turn out to be

$$\left[\begin{array}{cc|c}1&1&1\\2&3&c\\4&9&c^2\end{array}\right]$$

$R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 4R_1$

$$\sim\left[\begin{array}{cc|c}1&1&1\\0&1&c-2\\0&5&c^2-4\end{array}\right]$$

$R_1 \to R_1 - R_2$ and $R_3 \to R_3 - 5R_2$

$$\sim\left[\begin{array}{cc|c}1&1&1\\0&1&c-2\\0&0&c^2-5c+6\end{array}\right]$$

$$\sim\left[\begin{array}{cc|c}1&1&1\\0&1&c-2\\0&0&(c-2)(c-3)\end{array}\right]$$

so this system is consistent if $c=2$ or $c=3$

4. Convert the following matrices into the rref

$$\begin{bmatrix}2&4&10&-18\\-1&-2&-1&3\\-2&-3&0&3\\1&1&-1&-5\end{bmatrix},\ \begin{bmatrix}0&3&-6&6&4&-5\\3&-7&8&-5&8&9\\3&-9&12&-9&6&15\end{bmatrix}$$

**Solution:** a)

$$A=\begin{bmatrix}2&4&10&-18\\-1&-2&-1&3\\-2&-3&0&3\\1&1&-1&-5\end{bmatrix}$$

$R_3 \to R_3 + R_1, R_4 \to R_4 + R_2$

$$A \sim \begin{bmatrix} 2 & 4 & 10 & -18 \\ -1 & -2 & -1 & 3 \\ 0 & 1 & 10 & -15 \\ 0 & -1 & -2 & -2 \end{bmatrix}$$

$R_2 \to R_2 + R_1/2, R_4 \to R_4 + R_3, R_1 \to R_1/2$

$$A \sim \begin{bmatrix} 1 & 2 & 5 & -9 \\ 0 & 0 & 5 & -6 \\ 0 & 1 & 10 & -15 \\ 0 & 0 & 8 & -17 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3, R_3 \to R_3 - 2R_2, R_4 \to 5R_4 - 8R_2$

$$A \sim \begin{bmatrix} 1 & 0 & -15 & 21 \\ 0 & 0 & 5 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & -37 \end{bmatrix}$$

$R_1 \to R_1 + 3R_2, R_4 \to -R_4/37$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 5 & -6 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 3R_4, R_2 \to R_2 + 6R_4, R_3 \to R_3 + 3R_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2/5$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b)

$$B = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$B \sim \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & -2 & 4 & -4 & -2 & 6 \end{bmatrix}$$

$R_3 \to 3R_3 + 2R_1, R_2 \to 3R_2 + 7R_1$

$$B \sim \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & 0 & -18 & 27 & 52 & -8 \\ 0 & 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$

$$B \sim \begin{bmatrix} 0 & 3 & -6 & 6 & 0 & -21 \\ 3 & 0 & -18 & 27 & 52 & -8 \\ 0 & 0 & 0 & 0 & 2 & 8 \end{bmatrix}$$

$R_1 \to R_1/3, R_2 \to R_2/3, R_3 \to R_3/3$

$$B \sim \begin{bmatrix} 0 & 1 & -2 & 2 & 0 & -7 \\ 1 & 0 & -6 & 9 & 52/3 & -8/3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_2 \to R_2 - 52R_3/3$

$$B \sim \begin{bmatrix} 0 & 1 & -2 & 2 & 0 & -7 \\ 1 & 0 & -6 & 9 & 0 & -72 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$B \sim \begin{bmatrix} 1 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

5. Evaluate the rank of matrices which gives in the problem (5).

**Solution:** a)

$$A \sim \begin{bmatrix} \textcircled{1} & 0 & 0 & 0 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

Since there are 4 pivots, the rank of matrix $A$ is 4. b)

$$B \sim \begin{bmatrix} \textcircled{1} & 0 & -6 & 9 & 0 & -72 \\ 0 & \textcircled{1} & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 4 \end{bmatrix}$$

Since there are 3 pivots, the rank of matrix $B$ is 3.

6. Reduce the following matrix into rref (Row-Reduced Echelon form) and find its rank

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 14 \end{bmatrix}$$

Also list the pivotal elements of the matrix.

**Solution:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 14 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$R_1 \to R_1 - 3R_2, R_3 \to R_3 - 5R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} \textcircled{1} & 0 & 0 \\ 0 & \textcircled{1} & 0 \\ 0 & 0 & \textcircled{1} \end{bmatrix}$$

There are 3 pivot elements in $A$ which are shown by the circles, hence the rank of the matrix $A$ is 3.

7. Consider the transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Is this transformation linear. If so, find its matrix representation.

**Solution:**

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$$\begin{aligned} T[X+Y] &= T\left[\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right] \\ &= T\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix} \\ &= (x_1+y_1)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (x_2+y_2)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \\ &= \left(x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) + \left(x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) \\ &= T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + T\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{aligned}$$

Now let, $\alpha$ be a scalar:

$$\begin{aligned} T\left(\alpha\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= T\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \end{bmatrix} \\ &= \alpha x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \alpha x_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \\ &= \alpha\left(x_1\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right) \\ &= \alpha T\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Hence, it is a linear transformation. Consider the ordered basis of $\mathbb{R}^2$

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ and } T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Therefore the matrix representation of $T$ is:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$