Examples of fixed pt. iteration method.

Q1) Find a solution of $f(x) = x^3 + x - 1 = 0$ by iteration.

Ans) We require a form: $x = g(x)$

so rewrite $x^3 + x - 1 = 0$

$$\Rightarrow x(1+x^2) = 1$$

$$\Rightarrow x = \frac{1}{1+x^2} \equiv g_1(x)$$

Does this work to set up a fixed pt. iteration scheme?

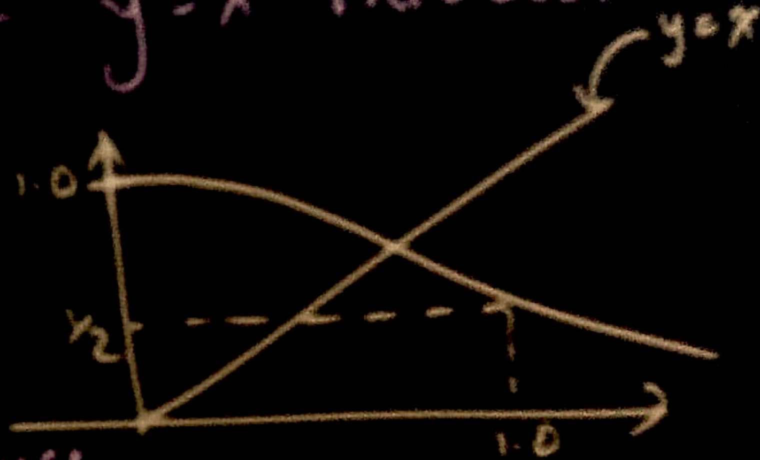
Check $g_1'(x) = \frac{2x}{(1+x^2)^2} \Rightarrow |g_1'(x)| = \frac{2|x|}{(1+x^2)^2} < 1 \ \forall x$ (why?)

b/c $(1+x^2)^2 = 1 + 2x^2 + x^4$
$= 1 + 2|x||x| + x^4 > 2|x|$

Next, where should we begin? $x_0 = ?$

Recall we are solving for $x = g_1(x)$ so the root of $f(x)$ lies where $y = x$ intersects w/ the curve $y = g_1(x)$

Do a rough sketch!



So choose $x_0 = 1$ & convergence is guaranteed to a unique fixed pt.!

Set up an iteration $x_{n+1} = \dfrac{1}{1+x_n^2}$

$x_1 = \dfrac{1}{1+x_0^2} = \dfrac{1}{1+1} = 0.5$; $x_2 = \dfrac{1}{1+x_1^2} = 0.8$,

$x_3 = \dfrac{1}{1+x_2^2} = 0.610$, $x_4 = \dfrac{1}{1+x_3^2} = 0.729$; $x_5 = 0.653$;

(3)

$x_6 = 0.701 \ldots$ the root exact to 6 dec. places

is $s = 0.682328$.

* Ques) Recall we are finding a root of $f(x) = x^3 + x - 1 = 0$

We could have chosen

$x = 1 - x^3 = g_2(x)$ & set up

an iteration scheme like below:-

$$x_{n+1} = 1 - x_n^3$$

Will this work?

Answer is No!

B/c $|g_2'(x)| = 3x^2 > 1$ near the root ≈ 0.63

You may check this by considering

$$x_{n+1} = 1 - x_n^3 \quad \text{w/} \quad x_0 = 1$$

$$x_1 = 1 - 1 = 0$$
$$x_2 = 1 - 0 = 1$$
$$x_3 = 1 - x_2^3 = 0$$
$$x_4 = 1 - x_3^3 = 1$$
$$x_5 = 1 - 1 = 0$$

the iterates will simply oscillate b/n 0 and 1.

Q2) Let us try to find $\sqrt{5} = ?$ using fixed pt. iteration scheme.

Ans) Equivalently we must be solving for $x^2 = 5$ or $x^2 - 5 = 0$.

Trials: set up $x_{n+1} = g(x_n)$

T1	T2	T3	T4
$g_1(x) = x - x^2 + 5$	$g_2(x) = \frac{5}{x}$	$g_3(x) = 1 + x - \frac{x^2}{5}$	$g_4(x) = \frac{1}{2}\left(x + \frac{5}{x}\right)$

(Where should we begin? $x_0 = 2.5$ is reasonable!

α is the root ($\sqrt{5}$)

n	T1 $g_1(x) = x - x^2 + 5$	T2 $g_2(x) = \frac{5}{x}$	T3 $g_3(x) = 1 + x - \frac{x^2}{5}$	T4 $g_4(x) = \frac{1}{2}\left(x + \frac{5}{x}\right)$
0	2.5	2.5	2.5	2.5
1	1.25	2.0	2.25	2 25
2	4.6875	2.5	2.2375	2.2361
3	-12.2852	2.0	2.2362	2.2361
Is it working	No	No	Yes	Yes
$g'(x)$	$1-2x$	$-\frac{5}{x^2}$	$\frac{1-2x}{5}$	$\frac{1-5/x^2}{2}$
$g'(\alpha)$	$1-2\sqrt{5}$ -3.47	-1	$= 0.11$	0
$g''(\alpha)$				0.44
			Lin conv	Quad. conv.

th^m: Assume that $g(x)$ is continuously differentiable in an interval I_α containing the f.p. α &

$$g'(\alpha) = g''(\alpha) = \cdots g^{(p-1)}(\alpha) = 0 \; ; \; p \geq 2$$

then for x_0 close enough to α;

$$x_n \longrightarrow \alpha$$

And $|\alpha - x_{n+1}| \leq K |\alpha - x_n|^p$

i.e. order of conv. is "p"

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