Lec-13: Change of Basis matrix to transform from one set of Bases to another set of basis of a given vector sp. 6/3/25

Let us consider $V = \mathbb{P}_2$

$W = \mathbb{P}_2$

1st we will consider the bases for V and W to be $B = \{1, x, x^2\}$

Before we discuss any linear transformation $T : \mathbb{P}_2 \to \mathbb{P}_2$.

let us see if we have a "change of Basis" matrix to move b/n the basis B & new basis of $\mathbb{P}_2$, $B' = \{1, 2x, 4x^2-2\}$

We will investigate this matter by first writing B' in terms of B.

$$1_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad (2x)_B = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}; \quad 4x^2-2 = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}$$

Here $B' \xrightarrow{S} B$ i.e. $(\Psi)_B = S(\Psi)_{B'}$

$$\text{Matrix trans}^n, S = A = \begin{pmatrix} | & | & | \\ T(1) & T(2x) & T(4x^2-2) \\ | & | & | \end{pmatrix}$$

$(\Psi)_{B'} = S^{-1}(\Psi)_B$

$$= \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \quad (1)$$

DATE 6 03 2025

The Bases of P_2:

$$B = \{1, x, x^2\} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$B' = \{1, 2x, 4x^2-2\} = \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

$$B' \xrightarrow{S} B$$

Now B' in terms of B

$1_{B'}$ $\quad$ 1_B

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \Longleftrightarrow \begin{pmatrix}1\\0\\0\end{pmatrix}$$

B'

$(2x)_{B'}$ $\quad$ $(2x)_B$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \Longleftrightarrow \begin{pmatrix}0\\2\\0\end{pmatrix}$$

$(4x^2-2)_{B'}$ $\quad$ $(4x^2-2)_B$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \Longleftrightarrow \begin{pmatrix}-2\\0\\4\end{pmatrix}$$

$$S = \begin{pmatrix}1 & 0 & -2\\0 & 2 & 0\\0 & 0 & 4\end{pmatrix}$$

DATE __/__/__

Let us write a polynomial in $\mathbb{P}_2$ $\varphi(x)$

basis $B' = \{1, 2x, 4x^2 - 2\}$

$(\varphi(x))_B = 7 + 8x - 12x^2$

$(\varphi(x))_{B'} = 1 + 4(2x) - 3(4x^2 - 2)$

$$(\varphi)_{B'} = \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}_{B'} \equiv \begin{pmatrix} 7 \\ 8 \\ -12 \end{pmatrix}_B$$

Let's check if we multiply

S to $\begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}_{B'}$, do we get $\begin{pmatrix} 7 \\ 8 \\ -12 \end{pmatrix}_B$

$$S(\varphi)_{B'} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix}_{B'}$$

$$= \begin{pmatrix} 7 \\ 8 \\ -12 \end{pmatrix}_B \checkmark$$

$\because$ the col^m vectors are linearly independent,

S is invertible.

$\therefore$ In order to move from

$$B \xrightarrow{S^{-1}} B'$$

the matrix transformation must be

$$S^{-1} = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

Q. Express $f(x) = 1 + 2x + x^2$ in the basis B'

$$(f)_{B'} = S^{-1}(f)_B = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}_B$$

$$= \begin{pmatrix} 3/2 \\ 1 \\ 1/4 \end{pmatrix}_{B'}$$

$$= \left(\frac{3}{2}\right)1 + 1(2x) + \frac{1}{4}(4x^2 - 2)$$

$$= \frac{3}{2} + 2x + x^2 - \frac{1}{2}$$

$$= 1 + 2x + x^2 \;\#$$

Change of bases for Linear transform^n

DATE __/__/__

Let $T : V \longrightarrow W$

Bases:- $\{B \underset{S}{\overset{S^{-1}}{\rightleftarrows}} B'\}$ $\qquad$ $\{C \underset{R}{\overset{R^{-1}}{\rightleftarrows}} C'\}$

two distinct set of basis vectors of V $\qquad$ Two distinct set of basis vectors of W

Also, A be the matrix representation of T w.r.t $B \to C$

Q) What is the matrix rep^n A' of T w.r.t. $B' \to C'$?

We have seen earlier,

$\forall \vec{v} \in V$ $\quad$ & $\quad$ $\forall \vec{w} \in W$

$$(\vec{v})_{B'} = S^{-1}(\vec{v})_B \qquad (w)_{C'} = R^{-1}(w)_C$$

$$T((\vec{v})_B) = A((\vec{v})_B)$$

$$\| $$

$$(T(\vec{v}))_C \; [redacted] = AS(\vec{v})_B$$

$$\|$$

$$R\,(T(\vec{v}))_{C'}$$

$$(T(\vec{v}))_{C'} = R^{-1}AS(\vec{v})_{B'} = A'(\vec{v})_{B'}$$

$$T((\vec{v})_{B'}) = A(\vec{v})_{B'} = AS^{-1}(\vec{v})_B$$

$$\|$$

$$(T(\vec{v}))_{C'}$$

$$\|$$

$$R^{-1}(T(\vec{v}))_C$$

$$(T(\vec{v}))_C = R\,AS^{-1}(\vec{v})_B$$

i.e. the matrix repⁿ of T w.r.t. $B' \to C'$

is $\boxed{A' = R^{-1} A S}$,

where R is the change of bases matrix from $C \to C'$ for W;

and

S is the change of bases matrix from $B \to B'$ for V.

Spl. Case

Linear Operator ($W \equiv V$)

$$T : \underset{\{B, B'\}}{V} \longrightarrow \underset{\{B, B'\}}{V}$$

then $\boxed{A' = S^{-1} A S}$

$T : \mathbb{P}_2 \to \mathbb{P}_2$

eg. $T(f) = f'$

DATE / /

$$\underset{(B \to B)}{A} = \begin{pmatrix} | & | & | \\ T(1) & T(x) & T(x^2) \\ | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$B = \{1, x, x^2\}$

$B' = \{1, 2x, 4x^2 - 2\}$

Recall

$$B' \xrightarrow{S} B \qquad S = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$B \xrightarrow{S^{-1}} B' \qquad S^{-1} = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

In this case

$\underset{(B' \to B')}{A'} =$ [redacted] $S^{-1}AS$

$$= \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 8 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{c} 1 \\ (2x) \\ (4x^2-2) \end{array} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} | & | & | \\ T(1) & T(2x) & T(4x^2-2) \\ | & | & | \end{pmatrix}$$