Eigenvalues & Eigenvectors

2.1 Agenda Items

- *Eigenvalues (evs) and Eigenvectors (EVs) of a matrix.*
- *Meaning of evs and EVs.*
- *Diagonalizable matrices and similar transformations.*
- *Analytical (pen-paper) method of finding evs.*
- *computational method of finding evs of a matrix(power method, etc).*

Definition 8 (evs and EVs)**.** Let $A \in \mathbf{M}_{n\times n}(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. A nonzero vector $x \in \mathbb{F}^n$ is an EV of A if $Ax = \lambda x$ for some $\lambda \in \mathbb{F}$. λ is said to be an ev A corresponding to the EV x.

2.2 Meaning of the equation $AX = \lambda x$

2.2.1 Algebraic meaning

$Ax = \lambda x$ can also be written as $(A - \lambda I)x = 0$, i.e., $\ker(A - \lambda I) = EVs \sqcup \{0\}$*. In the above equation we use that* $\lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$*. The subspace* $\ker(A - \lambda I)$ *has a special name,* EIGENSPACE of λ w.r.t. A.

Consider an example: $A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$ so that $A - \lambda I = \begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix}$. Now solving $Ax = \lambda x$ is equivalent to solving the system of linear equations $\begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This implies that

$$\begin{aligned} (2-\lambda)x_1 - x_2 &= 0 \\ 2x_1 + (4-\lambda)x_2 &= 0 \end{aligned}$$

Since EV cannot be 0, finding Evs of A boils down to the following question:

When does this system of linear equations have a nontrivial solution?

To answer the above question we need to know various features of an invertible matrices:

Let $B \in \mathbf{M}_{n\times n}(\mathbb{F})$, where $F = \mathbb{R}$ or $\mathbb{C}$. TFAE

- B is invertible.
- $Bx = b$ has a unique solution in $\mathbb{F}^n$ for all $b \in \mathbb{F}^n$.
- $\mathrm{rref}(B) = I_n$.
- $\mathbf{rank}(B) = n$.
- $\mathrm{im}(B) = \mathbb{F}^n$.
- $\ker(B) = \{0\}$.

Let us try to answer the above question now. In view of the above equivalence

$$\begin{aligned} \ker(A-\lambda I) \neq \{0\} &\iff (A-\lambda I) \text{ is not invertible} \\ &\iff \det(A-\lambda I) = 0. \end{aligned}$$

In our problem

$$\det(A-\lambda I) = \det \begin{pmatrix} 2-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} = (2-\lambda)(4-\lambda) + 2 = \lambda^2 - 6\lambda + 10$$

The above polynomial in λ is called the *characteristic polynomial for the matrix A. Thus*

$$\det(A - \lambda I) = 0 \iff \lambda = 3 \pm i$$

Let us call $\lambda_1 = 3 + i$ *and* $\lambda_2 = 3 - i$.

To find EV w.r.t. λ_1 *solve* $Ax = \lambda_1 x$. *After solving we obtain* $(1+i)x_1 + x_2 = 0$, *i.e.,* $x_2 = -(1+i)x_1$. *We can take* x_1 *to be any nonzero scalar of* $\mathbb{F}$, *say* k, *so as to write* $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k \\ -(1+i)k \end{pmatrix} = k \begin{pmatrix} 1 \\ -1-i \end{pmatrix}$. *Hence any nonzero multiple of the vector* $\begin{pmatrix} 1 \\ -1-i \end{pmatrix}$ *is an EV of the matrix A w.r.t the ev* λ_1. *Similarly one can find EV corresponding to the ev* λ_2.

2.3 A Slight Digression

Let $B \in \mathbf{M}_{n\times n}(\mathbb{F})$, *where* $\mathbb{F} = \mathbb{R}$ *or* $\mathbb{F} = \mathbb{C}$.

Question: Why $\textit{null}(B) = \{0\} \iff B$ *is invertible ?*

Answer: For finite dimensional vector spaces U, V *over* $\mathbb{F}$, *a linear transformation* $T : U \to V$ *is invertible if and only if* T *is one to one and onto.*
Rank Nullity Theorem:

$$\textit{nullity}(T) + \mathbf{rank}(T) = \dim(U).$$

Since T *is one to one* $\ker(T) = \{0\}$, *i.e.,* $\textit{nullity}(T) = 0$. *Also since* T *is onto,* $\mathbf{rank}(T) = \dim(V)$. *Therefore by Rank nullity theorem we obtain*

$$T \textit{ is an isomorphism} \implies \dim(U) = \dim(V).$$

2.4 HW/Exercise problem

Q. Consider $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}$

1. *Find the characteristic polynomial for A.*

2. *Find the evs of A.*

3. *Find the EVs of A*

Ans:

$$\begin{aligned} \textit{evs:} \quad & -1, 2, 3. \\ \textit{EVs:} \quad & \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \end{aligned}$$

Geometrical meaning of $Ax = \lambda x$, when λ is real. Ax is parallel to x, i.e., the "EV" x either gets stretched longitudinally when acted upon by the matrix A.

2.5 Coming Soon!

- *Diagonalizable matrix*
- *Similarity transformation*
- *Application of evs and EVs in solution to ODE*