Questions

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation defined by
$$T(x, y) = (3x + 2y, x + 4y).$$
Find its eigenvalues and eigenvectors.

2. Consider the linear transformation $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ (polynomials of degree at most 2) given by
$$T(f(x)) = xf'(x) + f(x).$$
Find its eigenvalues.

3. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by
$$T(x, y, z) = (y, z, x).$$
Find its eigenvalues.

4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by
$$T(x, y, z) = (x + y, y + z, z + x)$$
Find its eigenvalues and eigenvectors.

5. Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be defined by $T(A) = A^T$, where A^T is the transpose of A. Find the eigenvalues and eigenvectors of T.

6. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by the matrix
$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
Find its eigenvalues and eigenvectors.

7. Consider a transformation in $\mathbb{R}^2$ that rotates any vector counterclockwise by an angle θ.
   (a) Derive the standard matrix for this rotation by considering how the unit basis vectors transform.
   (b) Find the eigenvalues of this matrix.
   (c) Determine the eigenvectors corresponding to these eigenvalues.
   (d) Interpret the geometric meaning of the eigenvalues and eigenvectors in the context of rotation.

8. Show that the eigenvalues of the linear transformation given by the shear matrix

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

are always 1.

9. Let V be the plane in $\mathbb{R}^3$ given by the equation:

$$x_1 + 2x_2 + 3x_3 = 0.$$

Consider the linear transformation $T : V \to V$, and let the basis for V be:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ 1 \end{bmatrix}.$$

For each case below, determine the **eigenvalues** of the transformation matrix B (which represents T with respect to the given basis).

(a) T is the orthogonal projection onto the line spanned by $\mathbf{v}_1$.

(b) T is the orthogonal projection onto the line spanned by $\mathbf{w} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.