27/1/2019

Lecture (5) : Part (II)

Some more applications of analytic f^{n}s.

In the first part of Lecture (5), we saw that in case of simple (laminar) fluid flow we can find a complex velocity potential f^{n}, that is analytic, and captures all the information about the fluid flow eg., its velocity field, streamlines & velocity potential.

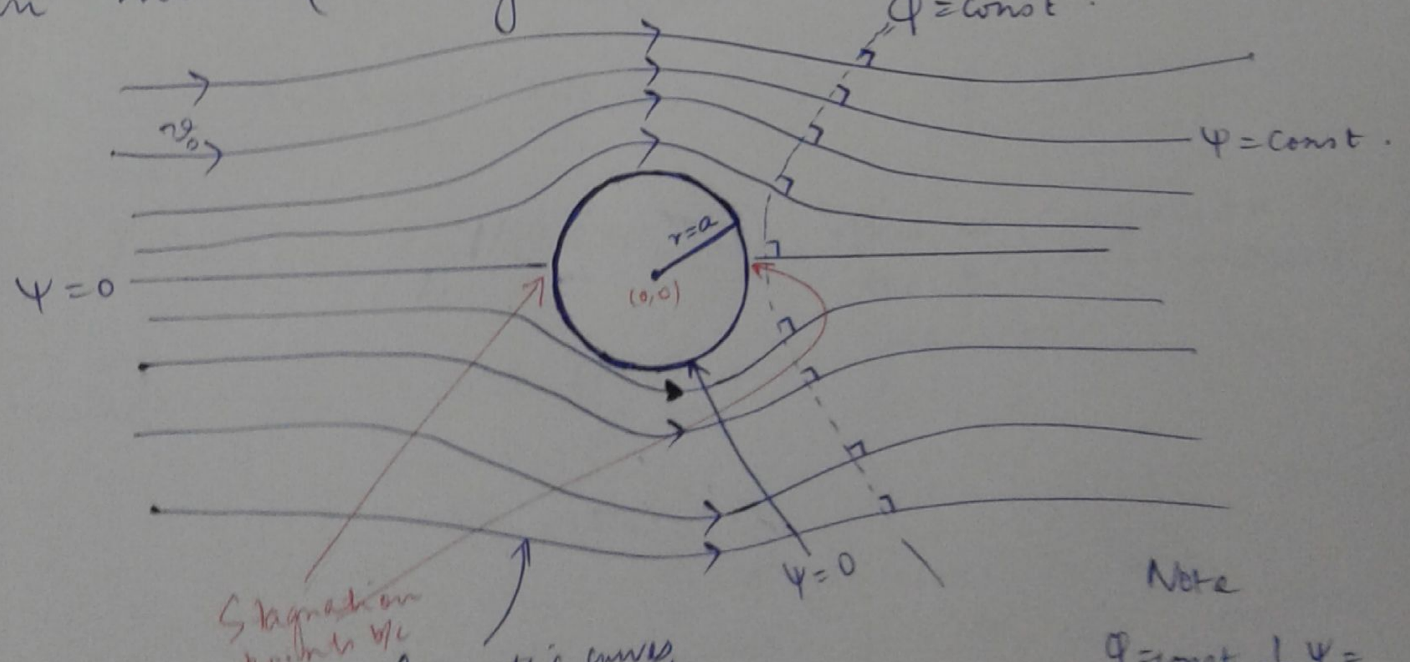
Fluid Flow around a Cylinder

Our goal here is to analyse the case when a cylinder is introduced in this fluid flow; specifically we want to find a way to capture all the information of the flow by a complex velocity potential f^{n}.

To this effect, let us consider the following complex velocity potential f^{n}.

This is merely an educated claim/guess $\rightarrow$ $$f(z) = v_0\left(z + \frac{a^2}{z}\right)$$; v_0, a are constants & $|z| > a$.

In fact we must have the following picture in mind (intuitively this should make sense)



Stagnation points b/c $\vec{v} = (0,0)$

Symmetric curves, even though the free hand drawing here does not give that impression.

Note

φ = const $\perp$ ψ = const

As found earlier pg ①

Recall, $f(z) = \varphi + i\psi = v_0\left(z + \frac{a^2}{z}\right)$ — ①

$$\xrightarrow{z = r(\cos\theta + i\sin\theta)} \underbrace{v_0\left(r + \frac{a^2}{r}\right)\cos\theta}_{\varphi(r,\theta)} + i\, v_0 \underbrace{\left(r - \frac{a^2}{r}\right)\sin\theta}_{\psi(r,\theta)}.$$

$$\Rightarrow f'(z) = \frac{df}{dz} = v_0\left(1 - \frac{a^2}{z^2}\right)$$

$\frac{dz^a}{dz} = a z^{a-1}$

$$= v_0\left(1 - \frac{a^2 e^{-2i\theta}}{r^2}\right)$$

$$= \frac{v_0}{r^2}\left\{r^2 - a^2(\cos 2\theta - i\sin 2\theta)\right\}$$

$$= v_0\left(1 - \frac{a^2\cos 2\theta}{r^2}\right) + i\,\frac{v_0 a^2 \sin 2\theta}{r^2}$$

Recall from the earlier example of fluid flow in part (1) of Lecture (5) that

To obtain this, we had used the Cauchy-Riemann eqs.

$\overline{f'(z)} = v_1 + i v_2$; where $\vec{v} = (v_1, v_2)$ — ②

$$\therefore \left.\begin{aligned} v_1 &= v_0\left(1 - \frac{a^2\cos 2\theta}{r^2}\right) \\ v_2 &= -v_0\,\frac{a^2\sin 2\theta}{r^2} \end{aligned}\right\} \quad \text{— ③}$$

Let us analyze eq. ③ & see if it makes sense

$\lim_{r\to\infty} (v_1, v_2) = (v_0, 0)$. This is compatible w/ the picture we drew earlier. ✓

Further, at $\begin{cases} r = a, \theta = 0 \text{ and} \\ r = a, \theta = \pi \end{cases}$

$\vec{v} = (0, 0)$

Hence these pts. are called <u>stagnation points</u>. ✓

Also, $r = a$, $\theta = 0$ and $\theta = \pi$ are streamlines b/c $\psi = \text{const}$. $\vec{v} = (0,0) = \left(\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}\right) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)$. ✓

Moreover,

$$\lim_{r\to\infty} \psi(r,\theta) = \lim_{r\to\infty} v_0\left(r-\frac{a^2}{r}\right)\sin\theta$$

$$= v_0 \underbrace{r\sin\theta}_{y} \quad \text{b/c } r \text{ dominates over } \frac{a^2}{r}$$

$$= v_0 y$$

& likewise

$$\lim_{r\to\infty} \varphi(r,\theta) = v_0 r\cos\theta = v_0 x$$

So for very large r (say R_∞) & correspondingly large y and x (say Y_∞ and X_∞), the streamline in (φ,ψ) plane is

$$\psi = v_0 Y_\infty = \text{const.} \quad (\text{Horizontal lines})$$

& potential $\varphi = v_0 X_\infty = \text{Const}$ (Vertical lines)

As is suggested by our intuitive picture earlier.

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