Ch(2): Root Finding Methods

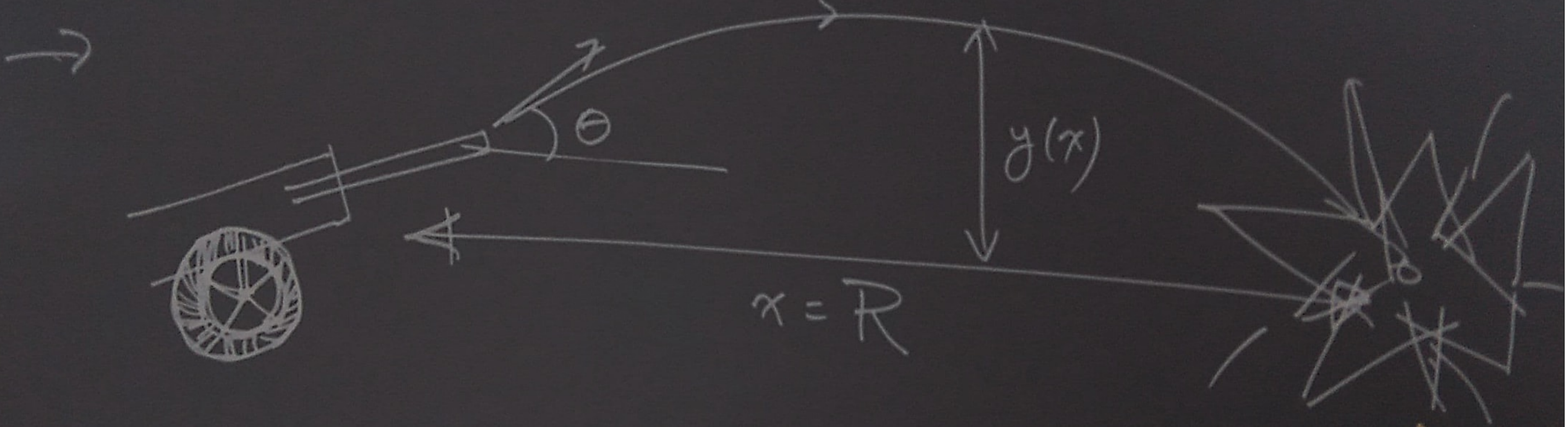
Problem statement :- Find such a value of x that makes $f(x) = 0$.

Why do we need root finding methods?

Many engineering & scientific applications

(Refer the compendium article on applications of root finding techniques on the course webpage).

One simple application of root finding methods.



Q) What angle θ should we use to force our cannon to hit a target a distance R away?

dist = speed × time

from Newton's laws of motion

Ans:- $x(t) = (v\cos\theta)t$; $y(t) = (v\sin\theta)t - \frac{1}{2}gt^2$

$t = \dfrac{x(t)}{v\cos\theta}$ plug it in here.

$$y(x(t)) = (\tan\theta)\,x - \frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2$$

When $x = R$ (shot hits the target)

$y(x) = 0$ b/c target is on the ground

but this is actually

$y(\theta) = 0$ w/ $x = R$

i.e. what value of θ makes $y(\theta) = 0$

So this is a root finding problem!

Of course the case is simple here, we do NOT need a ~~root~~ numerical root finding method but can solve this root finding problem by hand.

$y(x,\theta) = y(R,\theta) = 0.$

$\Rightarrow (\tan\theta)R - \frac{1}{2}g\left(\frac{R}{\cos\theta}\right)^2 \times \frac{1}{v^2} = 0.$

$\Rightarrow \frac{\sin\theta}{\cancel{\cos\theta}} R v^2 = \frac{1}{2} g \frac{R^2}{\cos\theta \cancel{\cos\theta}}$ b/c $\cos\theta \neq 0$ i.e. $\theta \neq \frac{\pi}{2}$

$\Rightarrow R = \frac{v^2}{g}\sin(2\theta)$

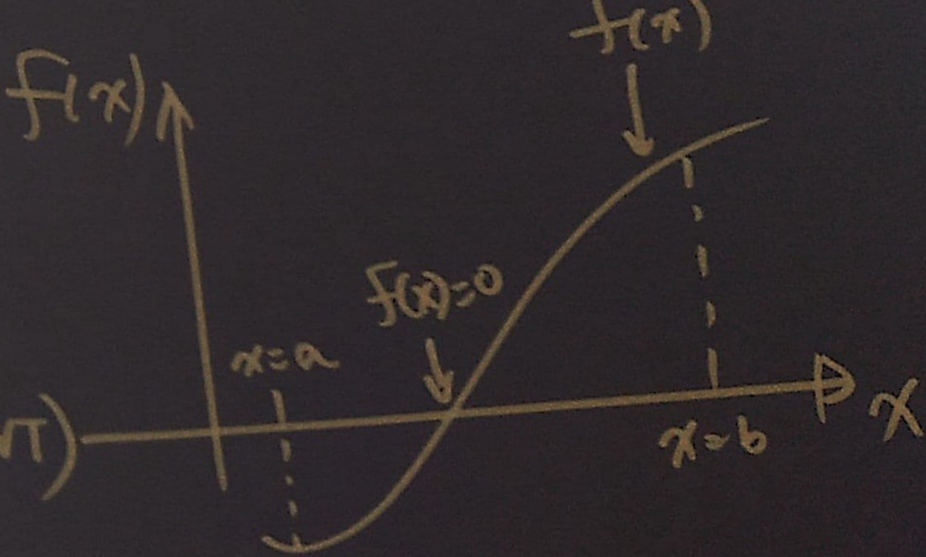
$\Rightarrow \theta = \frac{1}{2}\sin^{-1}\left(\frac{Rg}{v^2}\right)$ (v = vel. at which shot is fired).

But in most applications it will not be possible to analytically (by hand) solve for the roots of the f^n.

Several numerical techniques to find roots of f^n
→ Bisection method
→ Secant method
→ Newton's method
→ fixed pt. iteration method.
⋮
etc.

§(2.1) Bisection method

Intermediate Value th^m (IVT)



Let $f \in C[a,b]$

$\text{sign}\left(f(a) \times f(b)\right) < 0$ i.e. either $f(a) < 0$ & $f(b) > 0$
or
$f(a) > 0$ & $f(b) < 0$

then IVT guarantees the existence of atleast one such $x = p \in [a,b]$ whence $f(p) = 0$ i.e. p is a root of $f(x)$.

Convince yourself that this is indeed true by studying the figure in the previous slide!

Bisection method (looking for $x = p$ s.t. $f(p) = 0$)

i) Set $a_1 = a$ & $b_1 = b$; $p_1 = \dfrac{a_1 + b_1}{2}$ (mid pt.)

ii) If $f(p_1) = 0 \longrightarrow$ DONE! we have found the root.

else – do following steps:

iii) if $f(a_1) \times f(p_1) > 0$ then $p \in (p_1, b_1)$; set $a_2 = p_1$, $b_2 = b_1$

elseif $f(a_1) \times f(p_1) < 0$ then $p \in (a_1, p_1)$ set $a_2 = a_1$, $b_2 = p_1$

iv) Repeat steps (i) – (iii) w/ the interval $[a_2, b_2]$

* Apply steps (i) – (iv) if IVT holds

example

Find the root of $f(x) = x^3 + 4x^2 - 10 = 0$ in $[1, 2]$ by using the Bisection method.

Ans $f(1) = -5$; $f(2) = 14$ so IVT applies b/c f is $C[1,2]$ & $\exists\ p \in [1,2]$ s.t. $f(p) = 0$.

n	a_n	b_n	p_n	$f(p_n)$
1	1.0	2.0	1.5	2.375
2	1.0	1.5	1.25	−1.79687
3	1.25	1.5	1.375	0.16211
4	1.25	1.375	1.3125	−0.84839
5	1.3125	1.375	1.34375	−0.35098
6	1.34375	1.375	1.359375	−0.09641
⋮	⋮	⋮	⋮	⋮
13	1.364990235	1.365234375	1.365112305	−0.00194

$f(1) = -5$, $f(1.5) = 2.375$ so look for $p_2 \in [1, 1.5]$

$f(1.25) = -1.79687$, $f(1.5) = 2.375$ so look for $p_3 \in [1.25, 1.5]$

⋮

proceed so on!