Tutorial 1: Geometry of the Complex Plane

1. Write the following complex numbers in polar exponential form:

   (i) $-i$, (ii) $1-i$, (iii) $1-\sqrt{3}i$

2. Write the following complex numbers in Cartesian form:

   (i) $e^{2-i\frac{\pi}{2}}$, (ii) $\frac{1}{1-i}$, (iii) $\cos(c+i\frac{\pi}{4})$ where $c \in \mathbb{R}$.

3. Show that $z\overline{(z_1-z_2)} + z_1\overline{(z_2-z)} + z_2\overline{(z-z_1)} = 0$ is an equation of a straight line through z_1 and z_2.

4. Establish the following results:

   (i) $\mathcal{R}e(z) \leq |z|$, (ii) $|z_1z_2| = |z_1||z_2|$, (iii) $|w\bar{z}+\bar{w}z| \leq 2|wz|$

5. Sketch the following regions and state if they are *open*, *closed*, *bounded*, *compact* or *connected*.

   (i) $|2z+1+i| < 4$, (ii) $\mathcal{R}e(z) \geq 4$, (iii) $|z| \leq |z+1|$, (iv) $1 < |2z-1| \leq 2$, (v) $0 < arg(z) \leq \pi/2$, (vi) $\mathcal{R}e(z-z_0) > 0$ and $\mathcal{R}e(z-z_1) < 0$ for $z_0, z_1 \in \mathbb{C}$.

6. Show that $\mathcal{R}e\left(\frac{1-z^{n+1}}{1-z}\right) = 1 + r\cos\theta + r^2\cos 2\theta + ... + r^n\cos n\theta$ where $z = re^{i\theta}$.

7. Use the series representation of $exp(z)$: $e^z = \sum\limits_{j=0}^{\infty} \frac{z^j}{j!}$, $|z| < \infty$ to find the series representation of the functions: (i) $\sin z$, and (ii) $\sinh z$.

8. Use $e^{i(\theta_1+\theta_2)} = e^{i\theta_1}e^{i\theta_2}$ to deduce the following:

   (i) $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$, (ii) $\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v$.

9. Discuss the following mappings from the z-plane to the w-plane. Provide appropriate sketches to support your answers.
   (i) $w = z^3$, (ii) $w = \frac{1}{z}$.

10. Consider the transformation $w = z + \frac{1}{z}$, where $z = x+iy$ and $w = u+iv$.
    Show that $\mathcal{R} = \{z : y > 0 \text{ and } |z| > 1\}$ maps to $v > 0$.

11. To what curves on the sphere do the lines $\mathcal{R}e(z) = x = 0$ and $\mathcal{I}m(z) = y = 0$ correspond to?

12. Describe the curves on the sphere that correspond to any straight line on the z-plane.

13. The transformation $w = \frac{az+b}{cz+d}$, $(ad-bc) \neq 0$ is called a *bilinear* or *Möbius* transformation after the German geometer August Ferdinand Möbius. Here constants $a, b, c, d \in \mathbb{R}$ or $\mathbb{C}$.

    (a) Find the inverse map.
    (b) Find the fixed points of the map w.
    (c) Take $a = 1, b = z_0, c, d = 0$ and find the corresponding bilinear map.
    (d) Show that the application of a sequence of canonical transformations $w_1 = z + z_0, w_2 = bz$, and $w_3 = \frac{1}{z}$ can be used to synthesize the general bilinear transformation w defined above.

14. Find all bilinear mappings of the upper half plane ($\mathcal{I}m(z) \geq 0$) which maps into the unit disk $|w| \leq 1$.

15. Find a transformation which will map an infinite sector of angle $\pi/4$ into the interior of the unit circle.