# Solving Systems of Linear ODEs with Complex Eigenvalues

We present here the theory for a $2 \times 2$ system. (This can be generalized to a $n \times n$ system)

$$\vec{X}' = A_{2\times 2}\vec{X}$$

where the eigenvalues are $\lambda_{1,2} = \alpha \pm i\beta$

and the eigenvectors are $\vec{v}_1, \vec{v}_2 = \vec{p} \pm i\vec{q}$

Note that complex eigenvalues and eigenvectors always appear in pairs

We can then write the full solution as,

$$\vec{x}(t) = k_1 e^{\lambda_1 t}\vec{v}_1 + k_2 e^{\lambda_2 t}\vec{v}_2$$

However, since the $\lambda\,s$ and the $\vec{v}\,s$ are complex, we need to break up the solution space into real and imaginary parts to study the trajectories on the phase plane

To do this, we rewrite $\vec{x}(t)$ as –

$$\vec{x}(t) = \vec{x}_{re}(t) + i\vec{x}_{im}(t)$$

To see how this can be done, substitute $\lambda_1, \lambda_2, \vec{v}_1, \vec{v}_2$ in $\vec{x}(t) = k_1 e^{\lambda_1 t}\vec{v}_1 + k_2 e^{\lambda_2 t}\vec{v}_2$

Then,

$$\vec{x}(t) = k_1 e^{(\alpha+i\beta)t}(\vec{p}+i\vec{q}) + k_2 e^{(\alpha-i\beta)t}(\vec{p}-i\vec{q})$$
$$= k_1 e^{\alpha t} e^{i\beta t}(\vec{p}+i\vec{q}) + k_2 e^{\alpha t} e^{-i\beta t}(\vec{p}-i\vec{q})$$
$$= \underbrace{c_1 e^{\alpha t}(\vec{p}Cos\beta t - \vec{q}Sin\beta t)}_{\vec{x}_{re}(t)} + \underbrace{c_2 i e^{\alpha t}(\vec{p}Sin\beta t + \vec{q}Cos\beta t)}_{\vec{x}_{im}(t)}$$

$$e^{i\beta t} = Cos\beta t + iSin\beta t$$
$$e^{-i\beta t} = Cos\beta t - iSin\beta t$$
$$c_1 = k_1 + k_2$$
$$c_2 = k_1 - k_2$$

Therefore, $\vec{x}(t) = c_1\vec{x}_{re}(t) + c_2\vec{x}_{im}(t)$

Note that $c_2 i$ is rewritten as the new constant $c_2$.
We can do that as $i = \sqrt{-1}$ is also a constant

Question: Are $\vec{x}_{re}(t)$ and $\vec{x}_{im}(t)$ linearly independent solutions of $\vec{X}' = A\vec{X}$ ?

To check this, we substitute $\vec{x}(t) = \vec{x}_{re}(t) + i\vec{x}_{im}(t)$ in $\vec{X}' = A\vec{X}$

This gives, $\vec{x}'(t) = \underbrace{\vec{x}_{re}'(t)} + i\underbrace{\vec{x}_{im}'(t)} = \underbrace{A\vec{x}_{re}(t)} + i\underbrace{A\vec{x}_{im}(t)}$

Equating the real and the imaginary parts above, we get that both $\vec{x}_{re}(t)$ and $\vec{x}_{im}(t)$ satisfy the ODE, i.e. $\vec{x}_{re}'(t) = A\vec{x}_{re}(t)$ and $\vec{x}_{im}'(t) = A\vec{x}_{im}(t)$

Since $\vec{X}' = A\vec{X}$ is a $2 \times 2$ system, the two solutions $\vec{x}_{re}(t)$ and $\vec{x}_{im}(t)$ suffice and can be studied together on the phase-plane!

**Example:** Solve $\vec{X}' = A\vec{X}$ for $A = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$

Use $\begin{aligned} x' &= 6x - y \\ y' &= 5x + 4y \end{aligned}$ for Phase-Plane Trajectory

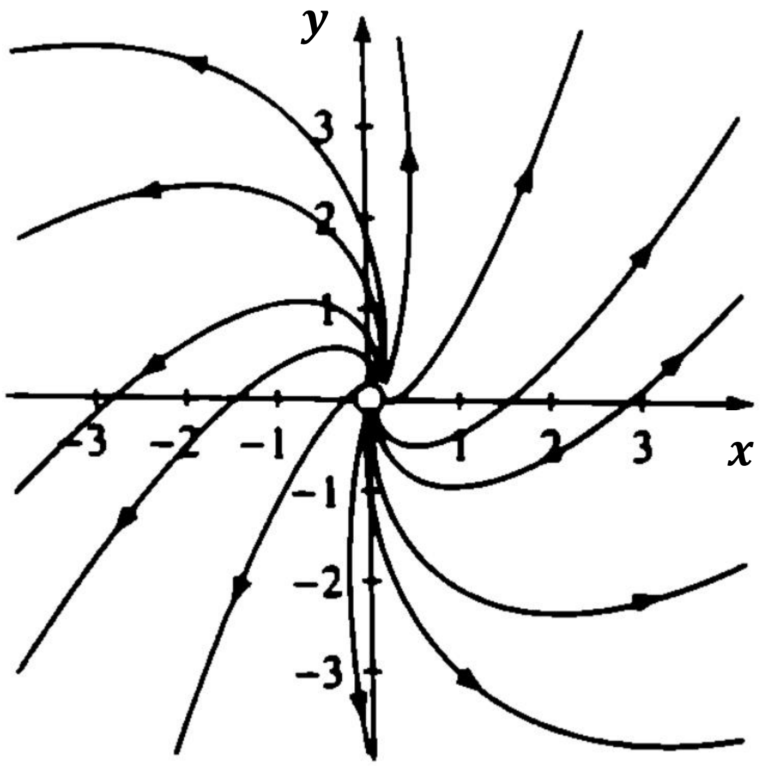
Eigenvalues of $A$: $\lambda_{1,2} = 5 \pm 2i$

Eigenvectors are: $\vec{v}_{1,2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

The corresponding general solution is

$$\vec{x}(t) = c_1\vec{x}_{re}(t) + c_2\vec{x}_{im}(t)$$
$$= e^{5t}\left\{ \begin{array}{l} c_1 \begin{pmatrix} Cos2t \\ Cos2t + 2Sin2t \end{pmatrix} \\ +c_2 \begin{pmatrix} Sin2t \\ Sin2t - 2Cos2t \end{pmatrix} \end{array} \right\}$$

where $c_1$ and $c_2$ are real constants




Phase-plane Trajectory for $\lambda_{1,2} = 5 \pm 2i$
(Note the unstable equilibrium at the origin)

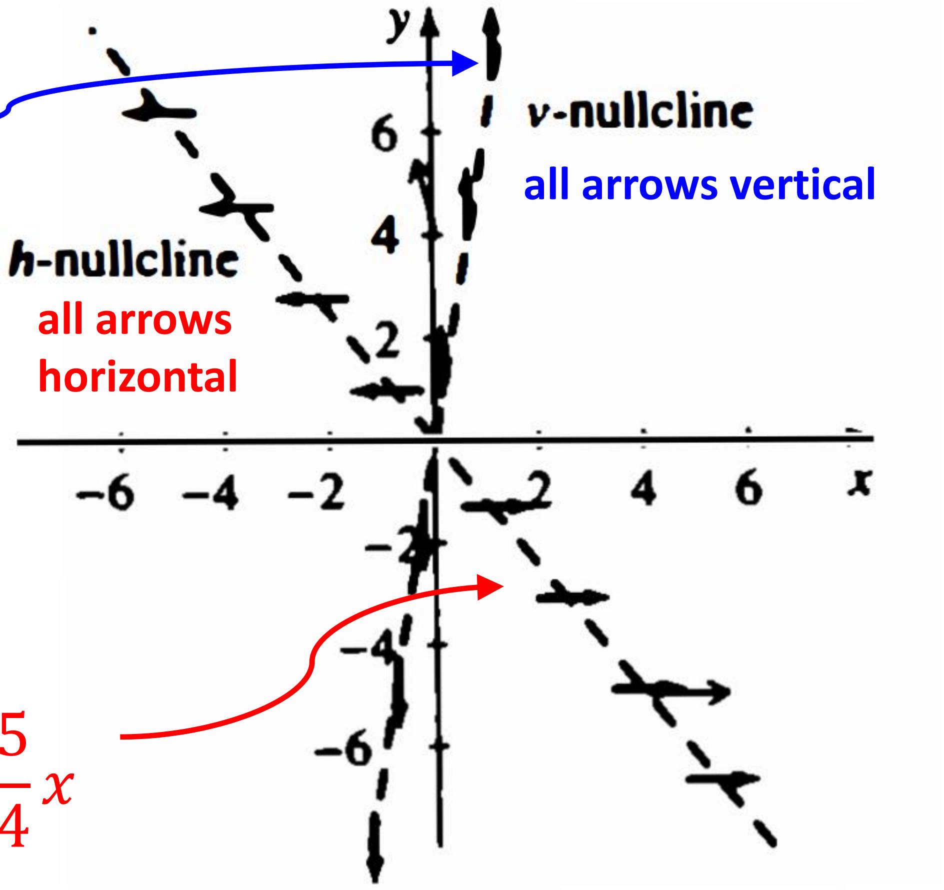
$\vec{X}' = A\vec{X}$ for $A = \begin{pmatrix} 6 & -1 \\ 5 & 4 \end{pmatrix}$ $\quad x' = 6x - y$, $\quad y' = 5x + 4y$

For $v - nullcline,$

$$x' = 0 \Rightarrow y = 6x$$

For $h - nullcline,$

$$y' = 0 \Rightarrow y = -\frac{5}{4}x$$



Example: Solve $\vec{X}' = A\vec{X} = \begin{pmatrix} 4 & -5 \\ 5 & -4 \end{pmatrix} \vec{X}$

Eigenvalues: $|A - \lambda I| = 0 \quad \Rightarrow \quad \lambda^2 + 9 = 0 \quad \Rightarrow \quad \lambda_{1,2} = \pm 3i$

Eigenvectors: $\vec{v}_{1,2} = \begin{pmatrix} 5 \\ 4 \mp 3i \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ -3 \end{pmatrix} = p + iq$ where $p = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ $q = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

Therefore,

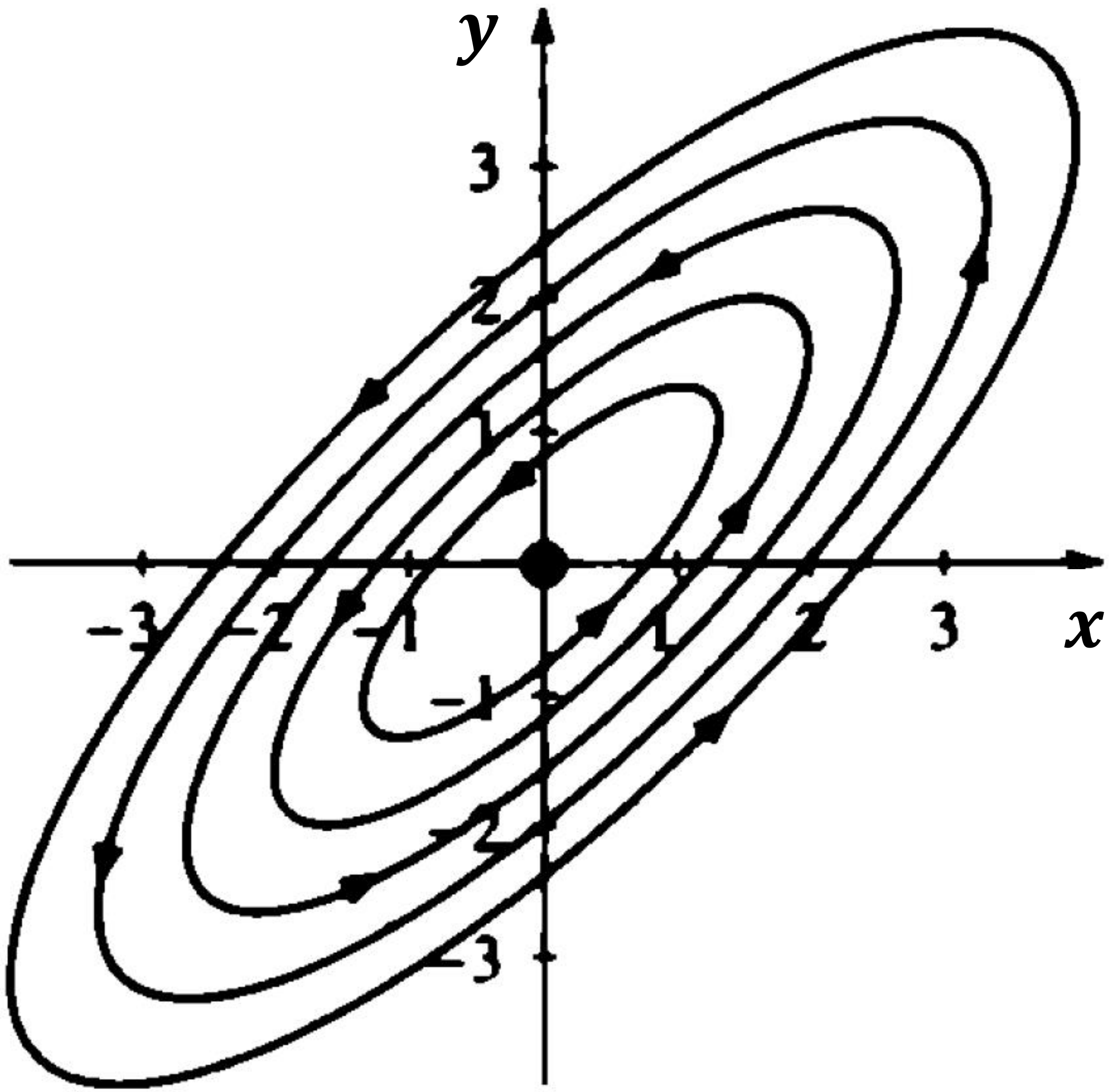
$$\vec{x}_{re}(t) = \cos 3t \begin{pmatrix} 5 \\ 4 \end{pmatrix} - \sin 3t \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

$$\vec{x}_{im}(t) = \sin 3t \begin{pmatrix} 5 \\ 4 \end{pmatrix} + \cos 3t \begin{pmatrix} 0 \\ -3 \end{pmatrix}$$

General Solution: $\vec{x} = c_1 \vec{x}_{re}(t) + c_2 \vec{x}_{im}(t)$

$$= c_1 \begin{pmatrix} 5\cos 3t \\ 4\cos 3t + 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 3t \\ 4\sin 3t - 3\cos 3t \end{pmatrix}$$

# Phase Portrait




* Note that the trajectories are really “Periodic Orbits” around the origin., i.e. a solution returns to the original point.

* The stable equilibrium at the origin neither attracts nor repels

* We see this kind of behavior when the roots are purely imaginary.

# Linear Independence of Functions over an interval $\boldsymbol{I}$

Suppose $f_1(t), f_2(t), \ldots \ldots .. f_n(t)$ are functions of $t$ on some interval $I$, such that they can be differentiated $n$ times on $I$.
We can then set up the following $n$ equations in $n$ unknowns using $n$ unknown constants $c_1, c_2, \ldots \ldots, c_n$ by successive differentiation for every $t$ in $I$.

$$c_1 f_1(t) + c_2 f_2(t) + \ldots\ldots\ldots\ldots + c_n f_n(t) = 0$$
$$c_1 f_1^{/}(t) + c_2 f_2^{/}(t) + \ldots\ldots\ldots\ldots + c_n f_n^{/}(t) = 0$$
$$\ldots\ldots$$
$$c_1 f_1^{(n-1)}(t) + c_2 f_2^{(n-1)}(t) + \ldots\ldots\ldots\ldots + c_n f_n^{(n-1)}(t) = 0$$

We know that if the determinant of the matrix coefficients of the $c_i$'s is not 0, then the only solution is the trivial one $c_1 = c_2 = \cdots \ldots = c_n = 0$ and the functions $f_1(t), f_2(t), \ldots \ldots .. f_n(t)$ are independent over the interval $I$.

$$W[f_1, f_2, \ldots, f_n](t) \equiv \begin{vmatrix} f_1(t) & f_2(t) & \ldots\ldots & f_4(t) \\ f_1^{/}(t) & f_2^{/}(t) & \ldots\ldots & f_2^{/}(t) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \ldots\ldots & f_n^{(n-1)}(t) \end{vmatrix}$$

**Wronskian of Functions**
$f_1(t), f_2(t), \ldots \ldots .. f_n(t)$
**on $\boldsymbol{I}$**

## The Wronskian and Linear Independence Theorem

If $W[f_1, f_2, \dots . f_n](t) \neq 0$ for all $t$ on the interval $I$, where $f_1, f_2, \dots . f_n$ are defined then $\{f_1, f_2, \dots . f_n\}$ is a set of linearly independent functions.

Note that if $\{f_1, f_2, \dots . f_n\}$ is linearly dependent on $I$ , then $W[f_1, f_2, \dots . f_n](t) \equiv 0$ on $I$ . So to show independence, we only need to find one $t_0 \in I$ such that $W[f_1, f_2, \dots . f_n](t_0) \neq 0$

$\Rightarrow$ linear independence at one point in $I$ implies independence over $I$

Example $\{t^2 + 1, t^2 - 1, 2t + 5\}$

$$W(t) = \begin{vmatrix} t^2+1 & t^2-1 & 2t+5 \\ 2t & 2t & 2 \\ 2 & 2 & 0 \end{vmatrix} = -8 \neq 0$$

Therefore, $\{t^2 + 1, t^2 - 1, 2t + 5\}$ is linearly independent over $t$ in $(-\infty, \infty)$

## Important: The Converse Is Not True!

Suppose that the Wronskian $W[f_1, f_2, \dots . f_n](t) = 0$ over an entire interval $I$, where $f_1, f_2, \dots . f_n$ are defined on $I$. Does this imply that $\{f_1, f_2, \dots . f_n\}$ is linearly dependent on $I$? **NO**

$$f_1(t) = t^3 \quad t \geq 0$$
$$= 0 \quad t < 0$$

$$f_2(t) = 0 \quad t \geq 0$$
$$= t^3 \quad t < 0$$

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = 0$$

However, it is directly evident that $f_1$ can never be a scalar multiple of $f_2$, so they are linearly independent and are not linearly dependent!

# Using the Wronskian to Establish Linear Independence for the Solutions of a Linear ODE

If $\vec{x}_1, \ldots\ldots \vec{x}_n$ solve a homogenous linear ODE system and if there exists any $t$ for which the Wronskian $W(\vec{x}_1, \ldots\ldots \vec{x}_n; t) \neq 0$ then $\vec{x}_1, \ldots\ldots \vec{x}_n$ are linearly independent solutions.

Here the Wronskian $W(\vec{x}_1, \ldots\ldots \vec{x}_n; t)$ is defined as -

$$W\left(\vec{x}_1, \ldots\ldots, \vec{x}_n; t\right) = \begin{vmatrix} | & \cdots & | \\ \vec{x}_1(t) & \cdots & \vec{x}_n(t) \\ | & \cdots & | \end{vmatrix}$$

Example

$$x' = 3x - 2y$$
$$y' = x$$
$$z' = -x + y + 3z$$

$$\Rightarrow \vec{X}' = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{pmatrix} \vec{X} \qquad \vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}; \ \vec{x}_2 = \begin{pmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}; \ \vec{x}_3 = \begin{pmatrix} e^{t} \\ e^{t} \\ 0 \end{pmatrix}$$

**Solutions**

We have three of them

but

are they independent?

$$W\left(\vec{x}_1, \vec{x}_2, \vec{x}_3; t\right) = \begin{vmatrix} 0 & 2e^{2t} & e^{t} \\ 0 & e^{2t} & e^{t} \\ e^{3t} & e^{2t} & 0 \end{vmatrix} = 2e^{2t}e^{3t}e^{t} - e^{t}e^{3t}e^{2t} = e^{6t} \neq 0$$

$\Rightarrow$ $\vec{x}_1(t), \vec{x}_2(t), \vec{x}_3(t)$ are linearly independent for any $t \in (-\infty, \infty)$

Therefore, the solution to this homogenous ODE is

$$\vec{X} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = c_1\begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix} + c_2\begin{pmatrix} e^{t} \\ e^{t} \\ 0 \end{pmatrix} + c_3\begin{pmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} c_2e^{t} + c_3(2)e^{2t} \\ c_2e^{t} + c_3e^{2t} \\ c_1e^{3t} + c_3e^{2t} \end{pmatrix}$$

$$\begin{aligned} x(t) &= c_2e^{t} + 2c_3e^{2t} \\ y(t) &= c_2e^{t} + c_3e^{2t} \\ z(t) &= c_1e^{3t} + c_3e^{2t} \end{aligned}$$

It would be interesting to solve this system using another approach! See the next slide!

$$x' = 3x - 2y$$
$$y' = x$$
$$z' = -x + y + 3z$$

Manipulate these algebraically to show that

$$z''' - 3z'' - 4z' + 12z = 0$$

Solution: $z(t) = c_1 e^{3t} + c_x e^{-2t} + c_3 e^{2t}$

Missing Term

$$\begin{aligned} x(t) &= & & c_2 e^{t} + & 2c_3 e^{2t} \\ y(t) &= & & c_2 e^{t} + & c_3 e^{2t} \\ z(t) &= & c_1 e^{3t} + & & c_3 e^{2t} \end{aligned}$$

Similarly, the first two equations can be manipulated to get

$$y'' - 3y' + 2y = 0$$

Solution: $y(t) = c_2 e^{t} + c_3 e^{2t}$

Now differentiate $y(t)$ to get $x(t)$

$$x(t) = y'(t)$$

Solution: $x(t) = c_2 e^{t} + 2c_3 e^{2t}$

Consider the solution $z(t) = k_1 e^{3t} + k_2 e^{-2t} + k_3 e^{2t}$ of $z''' - 3z'' - 4z' + 12z = 0$

Substituting this in $z' = -x + y + 3z$ we get $5k_2 e^{-2t} + k_3 e^{2t} = -x + y$

Note that, we also got $x(t) = c_2 e^{t} + 2c_3 e^{2t}$, $y(t) = c_2 e^{t} + c_3 e^{2t}$ (see previous slide)

$$\Rightarrow \qquad -x + y = -c_3 e^{2t}$$

Therefore, $5k_2 e^{-2t} + k_3 e^{2t} = -c_3 e^{2t}$

$\Rightarrow$ $k_1$ can be arbitrarily chosen (i.e. $k_1 = c_1$ as earlier), but $k_2$ and $k_3$ must satisfy the above equation

Comparing the coefficients of $\boldsymbol{e^{-2t}}$ and $\boldsymbol{e^{2t}}$ in the LHS and RHS of the above, we get $k_2 = 0, k_3 = -c_3$

Therefore, $z(t) = c_1 e^{3t} ++ c_3 e^{2t}$

Same solution as before!

So, "All Roads Do Indeed Lead to Rome"