Tutorial Worksheet (WL1.1, WL1.2 & WL2.1)
(Definition of vector spaces and its examples, Concept of linear dependence/independence vectors, Basis and dimension of vector spaces and its examples, properties of basis

Name and section: ______________________________

Instructor's name: ______________________________

1. Show that $V = \{(x, y, 0)|x, y \in \mathbb{R}\}$ form a vector space over the field $\mathbb{R}$.

**Solution:** We will prove that $V$ form a vector space.
Let $a, b, c \in \mathbb{R}$ and $u = (x_1, y_1, 0)$, $v = (x_2, y_2, 0)$, $w = (x_3, y_3, 0) \in V$, then

i. **Closure of Addition:-** if $u, v \in V$ then $u + v \in V$.
i.e $(x_1, y_1, 0) + (x_2, y_2, 0)$
$\Rightarrow (x_1 + x_2, y_1 + y_2, 0) \in V$
Hence, closure of addition property holds.

ii. **Closure of Scalar Multiplication:-** If $u \in V$ and $c \in \mathbb{R}$ then $c \cdot u \in V$.
i.e $c \cdot (x_1, y_1, 0)$
$\Rightarrow (cx_1, cy_1, 0) \in V$
Hence, closure of scalar multiplication property holds.

iii. **Commutativity of Addition:-** For all $u, v \in V$, $u + v = v + u$.
Let us take $u + v = (x_1, y_1, 0) + (x_2, y_2, 0) = (x_1 + x_2, y_1 + y_2, 0)$,
$= (x_2 + x_1, y_2 + y_1, 0) = (x_2, y_2, 0) + (x_1, y_1, 0) = v + u$.
Hence, commutativity of addition property holds.

iv. **Associativity of Addition:-** For all $u, v, w \in V$ , $(u + v) + w = u + (v + w)$.
Let us take $(u+v)+w = ((x_1, y_1, 0)+(x_2, y_2, 0))+(x_3, y_3, 0) = (x_1+x_2, y_1+y_2, 0)+(x_3, y_3, 0)$
i.e $= (x_1 + x_2 + x_3, y_1 + y_2 + y_3, 0)$,
Now, $u + (v + w) = (x_1, y_1, 0) + ((x_2, y_2, 0) + (x_3, y_3, 0)) = (x_1, y_1, 0) + ((x_2 + x_3, y_2 + y_3, 0))$
i.e $= ((x_1 + x_2 + x_3, y_1 + y_2 + y_3, 0))$
Clearly, $(u + v) + w = u + (v + w)$.
Hence, associativity of addition property holds.

v. **Additive Identity:-** There is an element called the zero vector and denoted 0 such that $u + 0 = u$ for all $u \in V$.
Let us take $u + 0 = (x_1, y_1, 0) + (0, 0, 0) = (x_1, y_1, 0) = u$.
Hence, (0, 0 , 0) is the additive identity of $V$.

vi. **Additive Inverse:-** For each element $u \in V$ there is an element $m \in V$ such that $u + m = 0$.
i.e. if $(x_1, y_1, 0) + (-x_1, -y_1, 0) = (0, 0, 0)$
$\Rightarrow (-x_1, -y_1, 0)$ is additive inverse of $(x_1, y_1, 0)$.
This $m$ is called the additive inverse of $u$ and is usually denoted $-u$.

vii. **Scalar Identity** For each $u \in V, 1 \cdot u = u$.
Here, $1 \cdot (x_1, y_1, 0) = (x_1, y_1, 0)$
$\Rightarrow$ 1 is the scalar identity of any vector $u \in V$.

viii. **Scalar Associativity:-** For all $u \in V$ and $a, b \in \mathbb{R}$, $(ab)u = a(bu)$.
Let us take, $(ab)u = (ab)(x_1, y_1, 0) = a(bx_1, bx_2, 0) = a(b(x_1, y_1, 0)) = a(bu)$.

ix. **Scalar Distribution:-** For all $u, v \in V$ and $a \in \mathbb{R}$, $a \cdot (u + v) = a \cdot u + a \cdot v$.
Let us take, $a \cdot (u + v) = a \cdot ((x_1, y_1, 0) + (x_2, y_2, 0))$
$= (ax_1, ay_1, 0) + (ax_2, ay_2, 0) = au + av$.

x. **Vector Distribution:-** For all $u \in V$ and $a, b \in \mathbb{R}$, $(a + b) \cdot u = a \cdot u + b \cdot u$.
Let $(a + b) \cdot u = (a + b)(x_1, y_1, 0)) = ((a + b)x_1 + (a + b)y_1, 0)$
$= (ax_1 + bx_1, ay_1 + by_1, 0) = ((ax_1, ay_1, 0) + (bx_1, by_1, 0)) = au + bu$.

2. Check whether it is vector space or not .

$$V = \{ax^2 + bx + c\ |a, b \in \mathbb{R} \text{ and } c = 1\}$$

**Solution:** Since there does not exist zero element in $V$ because $0 = 1$ not true.
hence $V$ is not a vector space.

3. Is the set $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ a linearly independent subset of $\mathbb{R}^2$.

**Solution:** let $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ consider

$$c_1 v_1 + c_2 v_2 = 0$$
$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

hence

$$c_1 = c_2 = 0$$

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ are linearly independent subset of $\mathbb{R}^2$.

4. Determine whether the given vectors are linearly independent or linearly dependent in $\mathbb{R}^3$

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

**Solution:**

In order to check linear independence let us consider the following linear relation

$$C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + C_3 \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ where } C_1, C_2, C_3 \text{ are scalars.}$$

This implies

$$\begin{bmatrix} C_1 + C_2 + C_3 \\ C_1 + 2C_2 + 3C_3 \\ C_1 + 3C_2 + 6C_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence we have the system of equations

$$C_1 + C_2 + C_3 = 0 \tag{1}$$
$$C_1 + 2C_2 + 3C_3 = 0 \tag{2}$$
$$C_1 + 3C_2 + 6C_3 = 0 \tag{3}$$

By equation 1 and 2, we obtain

$$C_2 + 2C_3 = 0.$$

Similarly by equation 1 and 3, we obtain

$$2C_2 + 5C_3 = 0.$$

From last two equations we obtain $C_2 = C_3 = 0$, which further implies $C_1 = 0$. Hence trivial solution is the only solution. This show that $\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \}$ is linearly independent in $\mathbb{R}^3$.

5. Check whether the following vectors forms a basis of $\mathbb{R}^2$ or not.

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$$

and if it is basis of $\mathbb{R}^2$ then write for any arbitrary vector $\begin{bmatrix} a \\ b \end{bmatrix}$ of $\mathbb{R}^2$ in the linear combination of $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \right\}$.

**Solution:** Consider the following linear relation

$$C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This implies that

$$\begin{bmatrix} C_1 + C_2 \\ 2C_1 + 4C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence we have the following equations

$$C_1 + C_2 = 0$$
$$2C_1 + 4C_2 = 0$$

Solving we obtain $C_1 = C_2 = 0$. Therefore $\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix}\}$ is a linearly independent set in $\mathbb{R}^2$. Since $\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$, hence $\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}1\\4\end{bmatrix}\}$ must be a basis for $\mathbb{R}^2$ over the field $\mathbb{R}$.
Let $x, y \in \mathbb{R}$ be such that

$$x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}1\\4\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

We obtain

$$x + y = a$$
$$2x + 4y = b$$

After solving these equations, we have the solutions of $x$ and $y$ in the form of $a$ and $b$ as

$$x = \frac{4a - b}{2} \text{ and } y = \frac{b - 2a}{2}.$$

6. Suppose $\{v_1, v_2, v_3, v_4\}$ are points lies on a line in $\mathbb{R}^2$ then show that set $\{v_2 - v_1, v_4 - v_3\}$ is linearly dependent subset of $\mathbb{R}^2$ .

**Solution:**
If $v_2 = v_1$ or $v_4 = v_3$ then we are done. So we assume that $v_2 \neq v_1$ and $v_4 \neq v_3$.

**Case-1:-**When the line parallel to y-axis.
then the equation of line of the form $x = m$
so,
$v_1 = (m, y_1),\ v_2 = (m, y_2),\ v_3 = (m, y_3)$ and $v_4 = (m, y_4)$
hence

$$v_2 - v_1 = (0, y_2 - y_1)$$
$$v_4 - v_3 = (0, y_4 - y_3)$$

and

$$(y_4 - y_3)(v_2 - v_1) - (y_2 - y_1)(v_4 - v_3) = 0$$

hence $\{v_2 - v_1, v_4 - v_3\}$ are linearly dependent.

**Case-2:-** When the line parallel to x-axis.
then the equation of line of the form $y = n$
so,

$v_1 = (x_1, n),\ v_2 = (x_2, n),\ v_3 = (x_3, n)$ and $v_4 = (x_4, n)$
hence

$$v_2 - v_1 = (x_2 - x_1, 0)$$

$$v_4 - v_3 = (x_4 - x_3, 0)$$

and

$$(x_4 - x_3)(v_2 - v_1) - (x_2 - x_1)(v_4 - v_3) = 0$$

hence $\{v_2 - v_1, v_4 - v_3\}$ are linearly dependent.
**Case-3:-** When the line passes through the origin .
then the equation of line of the form $y = px$
so,
$v_1 = (x_1, px_1),\ v_2 = (x_2, px_2),\ v_3 = (x_3, px_3)$ and $v_4 = (x_4, px_4)$
hence

$$v_2 - v_1 = (x_2 - x_1)(1, p)$$

$$v_4 - v_3 = (x_4 - x_3)(1, p)$$

and

$$(x_4 - x_3)(v_2 - v_1) - (x_2 - x_1)(v_4 - v_3) = 0$$

hence $\{v_2 - v_1, v_4 - v_3\}$ are linearly dependent.

**Case-4:-** When the line does not passes through origin .
then the equation of line of the form $y = ax + c$ where $a \neq 0$ and $c \neq 0$.
so,
$v_1 = (x_1, ax_1 + c),\ v_2 = (x_2, ax_2 + c),\ v_3 = (x_3, ax_3 + c)$ and $v_4 = (x_4, ax_4 + c)$
hence

$$v_2 - v_1 = (x_2 - x_1)(1, a)$$

$$v_4 - v_3 = (x_4 - x_3)(1, a)$$

and

$$(x_4 - x_3)(v_2 - v_1) - (x_2 - x_1)(v_4 - v_3) = 0$$

hence $\{v_2 - v_1, v_4 - v_3\}$ are linearly dependent.
In all the cases we get $\{v_2 - v_1, v_4 - v_3\}$ are linearly dependent subset of $\mathbb{R}^2$

7. let

$$W = \{(x, y, z) | x + y + z = 0\}$$

show that $W$ is a vector space over field $\mathbb{R}$ and find its basis and dimension.

**Solution:** It is sufficient to prove that $W$ is closed with respect to linear combinations, that is if $a(x_1, y_1, z_1)$ and $w(x_2, y_2, z_2)$ are in $W$ then also $a(x_1, y_1, z_1) + b(x_2, y_2, z_2) = (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2) \in W$ for all $a, b \in \mathbb{R}$. This is equivalent to say that if $x_1 + y_1 + z_1 = 0$ and $x_2 + y_2 + z_2 = 0$ then $(ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2) = 0$.
Now, we have $x + y + z = 0 \Rightarrow z = -x - y$.

Hence $(x, y, -x-y) = x(1,0,-1) + y(0,1,-1)$

so $\{(1,0,-1),(0,1,-1)\}$ are the basis of W and the dimension of $W$ is 2.