Advanced concepts on Markov processes.

Concepts → Hitting & return probabilities.

Concepts → Mean hitting (& return) times & absorption times.

(I) **Def^n (Hitting probabilities)**:- $\{X_n\}_{n\geq 0,\, n\in I}$ is a Markov chain w/ state space S.

Let $A \subset S$

T_A := first time the chain hits A starting from outside/inside A.

How shall we def^n T_A mathematically?

$$T_A := \min\{n \geq 0 \mid X_n \in A\}$$

w/ $T_A = 0$ if $X_0 \in A$

$T_A = \infty$ if $\{n\geq 0 \mid X_n \in A\} = \phi$

Often we are interested in calculating the following probability

$$g_k(l) = P(X_{T_A} = l \mid X_0 = k)$$

i.e. probability of hitting A through set A through state $l \in A$ starting from $k \in S$

k ← initial state

l ← final state

pg ②

Let us now find a formula for $g_k(l)$!

$S\backslash A = S - \{A\}$

Note $\forall k \in S\backslash A$; we have $T_A \geq 1$ given that $X_0 = k$

partitioning through (via) disjoint states $m \in S$

$$g_k(l) = P(X_{T_A} = l \mid X_0 = k) = \sum_{m \in S} P(X_{T_A} = l, X_1 = m \mid X_0 = k)$$

$$\underset{\text{prob}}{\overset{\text{cond}^n}{=}} \sum_{m \in S} P(X_{T_A} = l \mid X_1 = m, X_0 = k)\, P(X_1 = m \mid X_0 = k)$$

this step is similar to the steps in the Squirrel problem!

$$\underset{\text{property}}{\overset{\text{Markov}}{=}} \sum_{m \in S} P(X_{T_A} = l \mid X_1 = m)\, \underbrace{P(X_1 = m \mid X_0 = k)}_{p_{km}}$$

$$= \sum_{m \in S} p_{km}\, P(X_{T_A} = l \mid X_1 = m)$$

$$\boxed{g_k(l) = \sum_{m \in S} p_{km}\, g_m(l)} \quad ; \; k \in S\backslash A ;\; l \in A.$$

Defⁿ (Absorbing state):-

$$p_{kl} = I_{\{k=l\}} \quad \forall\, k, l \in A$$

i.e. $\{X_n\}$ is trapped (absorbed) in $A \subset S$

Note:- $\boxed{\sum_{l \in A} g_k(l) + P(T_A = \infty \mid X_0 = k) = 1}$

(II) Mean hitting times/Mean Absorption Times.

$h_k(A) := E(T_A \mid X_0 = k)$; clearly $h_k(A) = 0$ $\forall\, k \in A \subset S$.

Now let us derive a formula for $h_k(A)$ $\forall\, k \in S \setminus A$.

Again the steps will be similar to the squirrel problem!

$\forall\, k \in S \setminus A$;

$$h_k(A) = E(T_A \mid X_0 = k) = \sum_{m \in S} E(T_A, X_1 = m \mid X_0 = k)$$

Sum over partitioning events

$$= \sum_{m \in S} E(T_A \mid X_1 = m, X_0 = k)\, P(X_1 = m \mid X_0 = k)$$

Markov property

$$= \sum_{m \in A} \underbrace{E(T_A \mid X_1 = m)}_{1} p_{km} + \sum_{m \in S \setminus A} \underbrace{E(T_A \mid X_1 = m)}_{1 + h_m(A)} p_{km}$$

$$= \sum_{m \in A} p_{km} + \sum_{m \in S\setminus A} (1 + h_m(A))\, p_{km}$$

$$= \sum_{m \in S} p_{km} + \sum_{m \in S \setminus A} p_{km} h_m(A)$$

$$= 1 + \sum_{m \in S \setminus A} p_{km} h_m(A) + \sum_{m \in A} p_{km} h_m(A)$$

$$= 1 + \sum_{m \in S} p_{km} h_m(A)$$

$$\therefore \boxed{h_k(A) = 1 + \sum_{m \in S} p_{km} h_m(A)}$$

$\forall\, k \in S \setminus A$.

III First return time.

$$T_y^r := \min\{n \geq 1 \mid X_n = y\} \; ; y \in S.$$

w/ $T_y^r = \infty$ if $X_n \neq y \; \forall n \geq 1$.

Also note $T_y^r = T_y$ if $X_0 \neq y$.

Def^n (Mean return time to state y starting at x).

$$\mu_x(y) = E(T_y^r \mid X_0 = x) \geq 1$$

Similar derivation of formulae as in II !

$$\boxed{\mu_x(y) = 1 + \sum_{\substack{m \in S \\ m \neq y}} p_{xm}\, \mu_m(y)}$$

When $x = y$ then $\mu_y(y)$ is mean return time!