CTMC

Markov property for Continuous time Processes

$$P(X(t)=j \mid X(s)=i,\ X(t_{n-1})=i_{n-1},\ldots,X(t_1)=i_1)$$
$$= P(X(t)=j \mid X(s)=i); \text{ where } 0 \le t_1 \le t_2 \le \cdots \le t_{n-1} \le s \le t; \text{ and}$$

$i_1, i_2, \ldots, i_{n-1}, i, j \in S$ are $(n+1)$ states in the state space; $\forall\, n \ge 1,\ n \in I$.

This is called Markov Property (in continuous time).

Def^n (Continuous time Markov Chain or CTMC).

A continuous-time stochastic process $\{X(t) \mid t \ge 0\}$ is called a continuous-time markov chain (CTMC) if it has the Markov property.

→ Memorylessness (Markov property).

→ time-homogeneity.

Def^n (Time Homogeneity).

We say that a CTMC is time-homogeneous if for any $s \le t$ and any states $i, j \in S$

$$P(X(t)=j \mid X(s)=i) = P(X(t-s)=j \mid X(0)=i)$$
$$= P(X(t_1)=j \mid X(-t+s+t_1)=i)$$

& so on.

The key thing to note is that the difference in the time argument is $(t-s)$.

Not all CTMC need be time - homogeneous, but in this course we will only consider time - homogeneous CTMC!

Meaning of time-homogeneity.

Whenever the process enters state i; the way it evolves probabilistically from that pt is the same as if the process started in state i at time 0.

Def^n (Holding time) :- When the process enters state i, the time it spends there before it leaves state i is called the holding time.

T_i := holding time in state i
(just like in our motivating example of the n - server system).

Proposition :- $T_i \sim$ exponential D^n.

Recall; $p_{i,j}$ = probability of going from state i to j

Similar to Discrete time Markov chain

$q_{i,j}$ = rate at which the system goes from state i to j (rate at which the exponential alarm clocks go off).

Here, in CTMC, both $p_{i,j}$ & $q_{i,j}$ are f^n of time :- $p_{i,j}(t)$ & $q_{i,j}(t)$

$$p_{i,j} = \frac{q_{i,j}}{v_i} \quad ; \quad v_i = \sum_{j \in S} q_{i,j} < \infty$$

By def^n; $\boxed{q_{i,i} = 0}$

$$\Rightarrow q_{i,j} = v_i p_{i,j}$$

($v_i = 0 \Rightarrow$ state i is absorbing state)

The entries $p_{i,j}$ form a matrix $\mathbb{P} = \mathbb{P}(t)$ known as the Stochastic Matrix.

Note :- $q_{i,j}$ has more information about CTMC (continuous stochastic process) than $p_{i,j}$ b/c if we know all the $q_{i,j}$s then we can find v_i & $p_{i,j}$. But if we know the $p_{i,j}$s we cannot find $q_{i,j}$s!

Later we will see that the q_{ij}s form a matrix that is called the Generator matrix (generates the p_{ij}s).

$\therefore$ In many ways, $q_{i,j}$ are to CTMC what the $p_{i,j}$ are to DTMC.

$q_{i,j} > 0$ but $q_{i,j}$ need not be ≤ 1 like the $p_{i,j}$s.

Stochastic Matrix

$\mathbb{P}(t)$ comprise of $P_{ij}(t) = P(X(t)=j \mid X(0)=i)$

Note there is no "time step" in CTMC, instead we have $p_{ij}(t)$ which is a continuous f^n of time.

Often for CTMC; instead of writing $p_{i,j}(t)$ we use upper case $P_{i,j}(t) \equiv P_{ij}(t)$

eg if the CTMC is a poisson process; then

$$p_{ij}(t) = P\left(\text{there are } j-i \text{ events (arrivals) in } t \text{ interval of time}\right)$$

$$= \frac{(\lambda t)^{j-i} e^{-\lambda t}}{(j-i)!}$$

Chapman - Kolmogorove eqn. for CTMC.

$$p_{ij}(t+s) = \sum_{k \in S} p_{kj}(s)\, p_{ik}(t)$$

$$\equiv \sum_{k \in S} p_{ik}(t)\, p_{kj}(s).$$

the above is the $(ij)^{th}$ entry of $\mathbb{P}(t+s)$

$$\boxed{\mathbb{P}(t+s) = \mathbb{P}(t)\,\mathbb{P}(s)}$$

Recall $T_i \sim \exp(v_i)$; $v_i = \sum q_{ij}$

$\therefore f_{T_i}(h) = v_i e^{-v_i h}$

$\therefore P(T_i \le h) = 1 - e^{-v_i h}$

$\Rightarrow P(T_i > h) = e^{-v_i h}$

(cmp. w/ n-server motivating example)

$$\xrightarrow[\text{expand abt } h=0]{\text{Taylor}} 1 - v_i h + \frac{(v_i h)^2}{2!} - \cdots$$

$$= 1 - v_i h + \underbrace{o(h)}_{\to O(h^2)}$$

$\Rightarrow P(T_i \le h) = 1 - (1 - v_i h + o(h))$
$= v_i h + o(h)$

$\therefore P(\text{0 transitions by time } h \mid X(0) = i)$
$= P(T_i > h) = 1 - v_i h + o(h)$

$P(\text{Exactly 1 transition by time } h \mid X(0) = i)$
$= 1 - P(\text{0 transitions by time } h \mid X(0) = i)$
$= 1 - (1 - v_i h + o(h)) = v_i h + o(h)$.

& Likewise, it can be shown that - Try it out yourself

$P(\text{2 or more transitions by time } h \mid X_0 = i)$
$= o(h)$

Now;

$$P(X(h) = j \mid X(0) = i) = \underbrace{(v_i h + o(h))}_{\text{exactly 1 event occurred}} p_{ij} + P(\text{2 or more events}) \cdot p_{ij}$$

$$= v_i p_{ij} h + o(h) + o(h)$$

$$= v_i p_{ij} h + o(h) \quad \text{— (I)}$$

Similarly.

$$P(X(h) = i \mid X(0) = i) = 1 - v_i h + o(h) + o(h)$$

$$= 1 - v_i h + o(h) \quad \text{— (II)}$$

Further

$$p_{ij}(t+h) = P(X(t+h) = j \mid X(0) = i)$$

$$\overset{\text{Law of}}{\underset{\text{total prob.}}{=}} \sum_{k \in S} P(X(t+h) = j \mid X(h) = k, X(0) = i) \times P(X(h) = k \mid X(0) = i)$$

$$\overset{\text{Markov}}{\underset{\text{prop}}{=}} \sum_{k \in S} P(X(t+h) = j \mid X(h) = k) \cdot P(X(h) = k \mid X(0) = i)$$

$$\overset{\text{reset}}{\underset{\text{clk or time homogeneity}}{=}} \sum_{k \in S} P(X(t) = j \mid X(0) = k)\, p_{ik}(h)$$

$$= \sum_{k \in S} p_{kj}(t)\, p_{ik}(h) \quad \leftarrow \text{Basically Chapman Kolmogorov}$$

Using (I) & (II) above:

$$p_{ij}(t+h) = p_{ij}(t)(1 - v_i h + o(h)) + \sum_{k \neq i} p_{kj}(t)(v_i p_{ik} h + o(h))$$

Or equivalently

$$p_{ij}(t+h) - p_{ij}(t) = -v_i p_{ij}(t) h + \sum_{k \neq i} p_{kj}(t) v_i p_{ik} h + o(h)$$

Dividing by h; taking limit $h \to 0$ & using $v_i p_{ik} = q_{ik}$

$$\boxed{p'_{ij}(t) = \sum_{k \neq i} q_{ik} p_{kj}(t) - v_i p_{ij}(t)} \text{ --- (A.1)}$$

Or equivalently

$$\boxed{\left(\mathbb{P}'(t)\right)_{ij} = \left(\mathbb{G}\,\mathbb{P}(t)\right)_{ij}} \text{ --- (A.2)}$$

or

$$\boxed{\mathbb{P}'(t) = \mathbb{G}\,\mathbb{P}(t)} \text{ --- (A.3)}$$

(A.1), (A.2) & (A.3) are called Kolmogorov's Backward Eqns.

w/ the infinitesimal generator matrix $\mathbb{G}$ given by

$$\boxed{\begin{aligned} g_{ij} &= q_{ij} \quad \forall\, i \neq j \\ g_{ii} &= -v_i \end{aligned}}$$

with bdy condn $\mathbb{P}(0) = \mathbb{I}$ (Identity matrix)

Likewise, we have Kolmogorov's Fwd eqns

$$\mathbb{P}'(t) = \mathbb{P}(t)\,\mathbb{G} \text{ --- (B.1)}$$

Kolmogorov's Bkwd & fwd eqns, w/ the bdy condn $\mathbb{P}(0) = \mathbb{I}$, both have the same soln

$$\mathbb{P}(t) = e^{t\mathbb{G}} := \mathbb{I} + t\mathbb{G} + \frac{(t\mathbb{G})^2}{2!} + \cdots \quad \text{(C)}$$

∴ Even though we cannot normally obtain $\mathbb{P}(t)$ in a simple closed form

We can use eqn (C) to obtain a numerical approxn to $\mathbb{P}(t)$ if $|S|$ is finite by truncating the ∞ sum to a finite sum.

* the solution $\mathbb{P}(t) = e^{t\mathbb{G}}$ shows how basic the generator matrix $\mathbb{G}$ is to the properties of CTMC.

We will find, in the subsequent lectures, that $\mathbb{G}$ is also a key element for determining stny. Dn of CTMC.

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Stny Distributions.

Def^n:- Let $\{X(t) \mid t \geq 0\}$ be a CTMC w/ state space S, generator $\mathbb{G}$ & matrix of transition probability f^n $\mathbb{P}(t)$

An $|S|$-dimensional (row) vector $\vec{\pi} = (\pi_i)_{i \in S}$ w/ $\pi_i \geq 0 \ \forall i$ and $\sum_{i \in S} \pi_i = 1$ is said to be a Stationary Distribution if

$$\vec{\pi} = \vec{\pi}\, \mathbb{P}(t) \quad \forall\, t \geq 0.$$

Question:- How does the generator $\mathbb{G}$ relate to the definition of Stny. D^n?

Ans:- $\vec{\pi}$ is a Stny Distribution

$$\iff \vec{\pi} = \vec{\pi}\, \mathbb{P}(t), \ \forall\, t \geq 0$$

$$\underset{\mathbb{P} = e^{t\mathbb{G}}}{\iff} \vec{\pi} = \vec{\pi} \sum_{n=0}^{\infty} \frac{(t\mathbb{G})^n}{n!}, \ \forall\, t \geq 0$$

from prev. lecture on Kolmogorov eqns for CTMC

$$\iff \vec{\pi} - \vec{\pi} = \sum_{n=1}^{\infty} \frac{t^n}{n!} \vec{\pi}\, \mathbb{G}^n ; \ \forall\, t \geq 0$$

$$\iff 0 = \vec{\pi}\, \mathbb{G}^n \quad \forall\, n \geq 1$$

$$\iff \boxed{\vec{\pi}\, \mathbb{G} = 0}$$

Sum of +ve terms is zero if and only if each summand is zero.

therefore the condition $\vec{\pi} = \vec{\pi}\,\mathbb{P}(t) \;\forall\, t \geq 0$, that will be quite difficult to check, reduces to the much simpler condition

$\boxed{\vec{\pi}\, G = 0}$. Always note $\sum_{i \in S} \pi_i = 0$.

a set of $|S|$ linear eqns.

i.e. $\pi_j v_j = \underbrace{\sum_{i \neq j} \pi_i q_{ij}}$; $v_j = \sum_{i \in S} q_{ji}$

GLOBAL BALANCE EQNS

Physical interpretation.

π_j : long run proportion of time the process is in state j

v_j : rate of leaving state j when the process is in state j

$\pi_i q_{ij}$: long run rate of going from state i to state j

$\Rightarrow \pi_j v_j =$ long run rate of leaving state j.

$\Rightarrow \sum_{i \neq j} \pi_i q_{ij} =$ long run rate of going to state j

$\Rightarrow$ "the long run rate <u>out</u> of state j" = "long run rate <u>into</u> state j"

i.e. it is a statement of dynamic equilibrium

& the eqn. $\pi G = 0$ is also called the Global Balance eqn/Balance eqns

Detailed Balance eqns. (Also known as Local Balance)

For a continuous time Markov Ch (CTMC) w/ transition matrix Q; if π_i can be found s.t. for every pair of states i & j

$$\pi_i q_{ij} = \pi_j q_{ji} \quad \text{— (Detailed Balance condition)}$$

holds; then by summing over j; the global balance eqns are satisfied & $\vec{\pi}$ is stny. Dn of the process.

* If such a solution can be found the resulting eqns are usually much easier than directly solving the global balance eqns.

* A CTMC is reversible $\Leftrightarrow$ detailed balance holds for every (i, j)

Note:- The equivalent detailed balance for DTMC is

$$\pi_i p_{ij} = \pi_j p_{ji} \quad \forall\ i, j \text{ pairs}$$

Limiting probabilities -

for a CTMC $\{X(t) \mid t \geq 0\}$:

$$\lim_{t \to \infty} P_{ij}(t) = \lim_{t \to \infty} P\left(X(t) = j \mid X(0) = i\right)$$

$$\equiv \Pi_j$$

i.e. limiting probability $(t \to \infty)$ $\equiv$ Stny prob. D^n.

Application of Local (detailed) Balance Equations

Local balance :- $\Pi_i q_{ij} = \Pi_j q_{ji} \quad \forall i, j \in S, \; i \neq j$

there are ${}^{|S|}C_2$ such eqns.

but typically most of the eqns are trivially satisfied b/c $q_{ij} = q_{ji} = 0$

* Recall that for global balance (& Stny Cond) to exist; it is not necessary that local balance condn. always holds but if local balance does hold then surely global balance (& Stny) do hold.

** One quick way to check if local balance does **NOT** hold for stationarity is to check if there are any rates pg(7)

q_{ij} and q_{ji} s.t. $q_{ij} > 0$ and $q_{ji} = 0$ or $q_{ij} = 0$ & $q_{ji} > 0$.

(In these cases, the local balance route to investigating stationarity will be futile)!