Rate of convergence of iterative methods (1)

Def^n:- Let $\{p_n\}_{n\geq 0} \longrightarrow p$ w/ $p_n \neq p \ \forall n$.

If $\exists$ +ve constants λ and α s.t.

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda \ ;$$

then $\{p_n\}_{n\geq 0} \longrightarrow p$ with order of convergence α

and asymptotic error const λ

$\alpha = 1 \Rightarrow$ lin. convergence.

$\alpha = 2 \Rightarrow$ Quadratic convergence.

etc . . .

* The fixed pt. iteration scheme $p_n = g(p_{n-1})$ converges linearly, when $g'(p) \neq 0$

ref. pg. 77 & notes where I have shown this numerically.

* for fixed pt. iteration (as above), we have $g'(p) = 0$ but $g''(p) \neq 0$ & bdd. from above; then Convergence is Quadratic

(again see pg. 77 & numerical example from prev. Lec. set).

Multiple roots.

Def^n :- A sol^n. p of $f(x)=0$ is a zero of multiplicity m of f if for $x \neq p$ we can write

$$f(x) = (x-p)^m q(x) \; ; \quad \lim_{x \to p} q(x) \neq 0$$

* If f is a polynomial f^n; then above def^n translates to

$$f(x) = (x-p)^m q(x) \; ; \quad q(p) \neq 0$$

where $q(x)$ is a poly. f^n.

Q) How to easily identify f"s w/ simple roots ($m=1$) & multiple roots ($m>1$)?

Ans) (I) $f \in C^1[a,b]$ has a simple zero at p in $(a,b) \iff f(p)=0$ but $f'(p) \neq 0$

(II) $f \in C^m[a,b]$ has a zero of multiplicity m at p in $(a,b) \iff$

$$0 = f(p) = f'(p) = f''(p) = \cdots\cdots = f^{(m-1)}(p);$$

but $f^{(m)}(p) \neq 0$.

If p is a (root) simple zero of $f(x)$ then Newton's method conv. Quadratically

(5)

Q) What about order of convergence of Newton's method when applied to f^{n}s. w/ multiplicity $m>1$?

Ans)

$$\mu(x) := \frac{f(x)}{f'(x)} \quad ; \quad f(x) = (x-p)^m q(x)$$

$$= \frac{(x-p)^m q(x)}{m(x-p)^{m-1} q(x) + (x-p)^m q'(x)}$$

$$= (x-p) \underbrace{\frac{q(x)}{mq(x) + (x-p) q'(x)}}_{\psi(x)}$$

$\mu(x)$ also has a zero at p ; but $\psi(p) = \frac{1}{m} \neq 0$

$\Rightarrow$ p is a simple root of $\mu(x)$.

Now apply Newton's method to $\mu(x)$ b/c μ has a simple zero for which Newton's method is quadratically convergent!

$$g(x) = x - \frac{\mu(x)}{\mu'(x)}$$

$$= x - \frac{f(x)\,f'(x)}{(f'(x))^2 - f(x)\,f''(x)}$$

Calculating this will incur numerical cost!

Q) What is the order of conv. of the Secant method?

Hint :- Assume for secant method, following is true $|p_{n+1} - p| \approx C|p_n - p|\underbrace{|p_{n-1} - p|}_{|e_{n-1}|}$ for sufficiently large n

Ans :- $e_n := p_n - p$

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^\alpha} = \lambda > 0 \Rightarrow \text{for suff. large } n \quad |e_{n+1}| \approx \lambda |e_n|^\alpha$$

$$\therefore |e_n| \approx \lambda |e_{n-1}|^\alpha \text{ and } |e_{n-1}| = \frac{1}{\lambda^{1/\alpha}} |e_n|^{1/\alpha}$$

$$\therefore \lambda |e_n|^\alpha \approx |e_{n+1}| \approx C|e_n| \lambda^{-1/\alpha} |e_n|^{1/\alpha}$$

so $|e_n|^{\alpha} \approx c\lambda^{-\frac{1}{\alpha}-1} |e_n|^{1+\frac{1}{\alpha}}$

this means
since powers of $|e_n|$ on both sides
must match

$$\alpha = 1 + \frac{1}{\alpha} \Rightarrow \alpha = \frac{1+\sqrt{5}}{2}$$

this is the golden ratio!