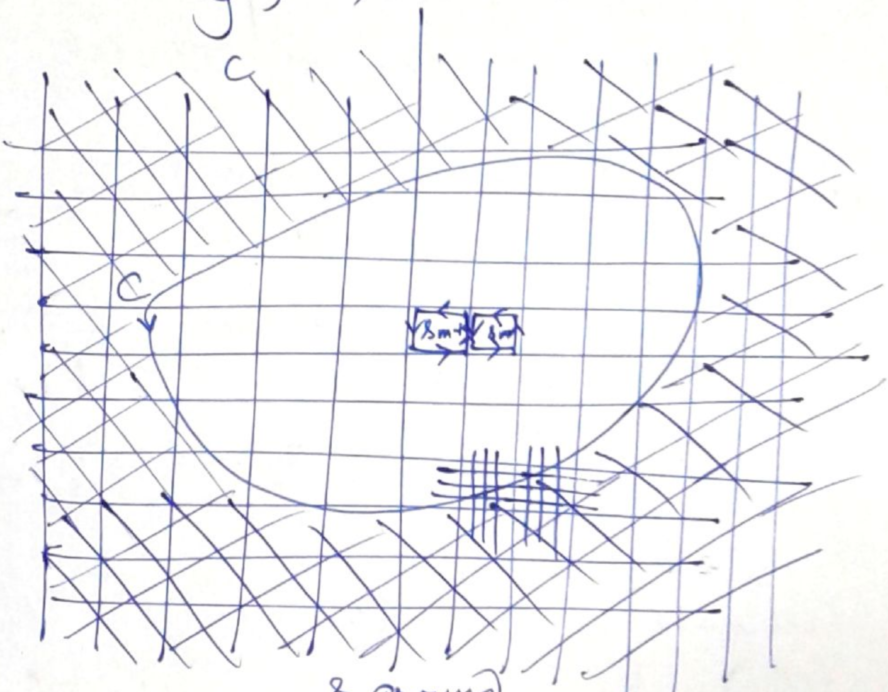
Lecture (9) :- Cauchy - Goursat Th^m (or simply Cauchy's Th^m)   13/2/19.

Th^m :- If $f(z)$ is analytic at all pts. w/in and on a simple closed contour, then

$$\oint f(z)\,dz = 0$$

Proof :-



(I) Overlay a mesh on & around the region enclosed by C. Consider those pts which are enclosed by C or on C and delete all other pts. The leftover pts. define the region R.

(II) Next, we refine the mesh & repeat the process in (I) above.

(III) The refinement process (II) is repeated until the length of the diagonal of each square cell is sufficiently small i.e. $\sqrt{2A_j} \ll 1$. Here A_j is the area of the jth cell.

Now we are left w/ a total of n square cells and partial square cells (bdy cells).

(IV) $\oint_C f(z)\,dz = \sum_{j=1}^{n} \oint_{C_j} f(z)\,dz$ where C_j is the closed contour around the j^{th} cell.

(V) Since we integrate w.r.t the variable z in (IV) along the contour C and C_j; we will consider the generic pt. z on the contour & z_j as some pt w/in or on the bdy of the cell C_j.

then re-writing $f(z)$ we have

$$f(z) = f(z_j) + (z-z_j)f'(z_j) + (z-z_j)\underbrace{\left\{\frac{f(z)-f(z_j)}{z-z_j} - f'(z_j)\right\}}_{\tilde{f}_j(z)}$$

$$= f(z_j) + (z-z_j)f'(z_j) + (z-z_j)\tilde{f}_j(z)$$

(VI) Now compute $\oint_{C_j} f(z)\,dz$ for $f(z)$ defined in (V)

$$\oint_{C_j} f(z)\,dz = \oint_{C_j} f(z_j)\,dz + \oint_{C_j}(z-z_j)f'(z_j)\,dz + \oint_{C_j}(z-z_j)\tilde{f}_j(z)\,dz$$

$$= f(z_j)\underbrace{\cancel{\oint_{C_j} dz}}_{0} + f'(z_j)\underbrace{\cancel{\oint_{C_j}(z-z_j)\,dz}}_{0} + \oint_{C_j}(z-z_j)\tilde{f}_j(z)\,dz$$

b/c of Thm (8.1) of Lecture 18)

$$\therefore \oint_{C_j} f(z)\,dz = \oint_{C_j}(z-z_j)\tilde{f}_j(z)\,dz$$

(VII) B/c the mesh is sufficiently refined we have

$$\lim_{z\to z_j} \frac{f(z)-f(z_j)}{z-z_j} = f'(z_j)$$

$\Rightarrow \exists\ \delta = \sqrt{2A_j}$ s.t. $|z - z_j| < \delta$ implies

$$\left| \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j) \right| < \epsilon$$ for some $\epsilon > 0$ small & chosen apriori.

$$\Rightarrow \left| \tilde{f}_j(z) \right| < \epsilon$$

(VIII) Now applying th^m (8.2) of ~~dete~~ lecture (8)

$$\left| \oint_C f(z)\,dz \right| \underset{\Delta\text{-ineq}}{\leq} \sum_{j=1}^{n} \left| \oint_{C_j} f(z)\,dz \right| = \sum_{j=1}^{n} \left| \oint_{C_j} (z - z_j)\tilde{f}_j(z)\,dz \right|$$

$$\leq \sum_{j=1}^{n} \oint_{C_j} |z - z_j| \left|\tilde{f}_j(z)\right| dz$$

$$\overset{Th^m (8.2)}{\leq} \begin{cases} \displaystyle\sum_{j=1}^{n_i} \underbrace{\sqrt{2A_j}\,\epsilon}_{M} \underbrace{4\sqrt{A_j}}_{L} \text{; for all interior cells} \\ + \\ \displaystyle\sum_{j=1}^{n_b} \underbrace{\sqrt{2A_j}\,\epsilon}_{M} \underbrace{(4\sqrt{A_j} + L_j)}_{L} \text{; for bdy cells} \end{cases}$$

Here n_i are the no. of interior cells & n_b are the no. of bdy cells.

(IX) Therefore,

$$\oint_C f(z)\,dz \leq \left(4\sqrt{2}A + \sqrt{2AL}\right)\epsilon$$

$\longrightarrow 0$ as ϵ is chosen sufficiently small.

Here $A = \sum_{j=1}^{n} A_j$ & $L = \sum_{j=1}^{n_b} L_j$ #