Diagonalizable Matrices

3.1 Agenda Item

- *Diagonalization of matrices*
- *Similarity transformation*
- *Spectral decomposition of matrices*

Last Lecture:

- *We define evs and EVs of a square matrix*
- *determinant and trace of a matrix and its relation with evs*

3.2 Diagonalizable Matrices

Certain forms of matrices are convenient to work with. For example

- *Upper/Lower triangular matrices*(*why?*)
- *Diagonal forms*(*why?*)

Think finding evs and powers of above matrices.

Wouldn't it be nice if

$$\begin{array}{cc} A \longrightarrow & D \\ (\textit{any } n \times n \textit{ matrix}) & (\textit{diagonal form}) \end{array}$$

$A \in \mathbf{M}_{n\times n}(\mathbb{F})$ is diagonalizable over $\mathbb{F}$ if there exists an invertible matrix S over $\mathbb{F}$ such that $A = SDS^{-1}$, or equivalently $D = S^{-1}AS$.

Note that the evs of A and D will be the same and the above relation $D = S^{-1}AS$ is known as the similarity transformation.

Q. When is a matrix diagonalizable?

Ans: $A \in \mathbf{M}_{n\times n}(\mathbb{F})$ is diagonalizable if and only if A has n linearly independent EVs in $\mathbb{F}^n$.

Note that an $n \times n$ complex matrix that has n distinct eigenvalues is diagonalizable.

Example 9. Q. Find a matrix that diagonalizes $A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$.

Ans: Solve $\det(A - \lambda I) = 0$ to obtain $\lambda_1 = 3 + i$ and $\lambda_2 = 3 - i$. Solving $Ax = \lambda_i x$ for $i = 1, 2$, we obtain

$$X_1 = \begin{pmatrix} 1 \\ -1-i \end{pmatrix}, X_2 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}$$

as Evs of A w.r.t. the evs λ_1, λ_2, respectively. We note that $S = \begin{pmatrix} 1 & 1 \\ -1-i & -1+i \end{pmatrix}$ diagonalizes A. Since

$$\begin{aligned} S^{-1}AS &= \begin{pmatrix} \frac{-1+i}{2i} & -\frac{1}{2i} \\ \frac{1+i}{2i} & \frac{1}{2i} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1-i & -1+i \end{pmatrix} \\ &= \begin{pmatrix} 3+i & 0 \\ 0 & 3-i \end{pmatrix} \\ &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ &= D \end{aligned}$$

The Column vectors of S form an eigenbasis for A and the diagonal entries of D are the associated evs.

Q. What are the evs and EVs of the $n \times n$ identity matrix I_n?

Is there an eigenbasis for I_n?

Which matrix diagonalizes I_n?

This is in some sense a silly and yet a conceptually trick question.

Example 10. Find the eigenspace of $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

The evs are given by 0 and 1 with algebraic multiplicity 1 and 2, respectively. To find EV consider

$$\begin{aligned} X_1 &= \ker(A - 1I) \\ &= \ker \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \ker \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathrm{sp} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \end{aligned}$$

where $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is the reduced row echelon form of the matrix $\begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. The calculation:

$$\begin{aligned} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \Longrightarrow \begin{pmatrix} x_2 \\ x_3 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

Above calculation shows that $x_2 = x_3 = 0$. Thus we can take any nonzero value as x_1 to obtain an EV of A w.r.t. the ev 1. For convenience we take $x_1 = 1$ to obtain $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ as an EV. Likewise $X_2 = \ker A = \text{sp} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Thus we are able to find only two linearly independent EVs. Hence we won't have an eigenbasis here, equivalently we cannot find S to diagonalize A.

3.3 Geometric multiplicity of ev

$$\begin{aligned} gemm(\lambda) &= \dim(\ker(A - \lambda I_n)) \\ &= nullity(A - \lambda I_n) \\ &= n - \mathbf{rank}(A - \lambda I_n) \end{aligned}$$

In previous example

$$gemm(1) = \dim(\ker(A - \lambda I_n)) = \dim\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle = 1 \neq almu(1) = 2.$$

Theorem 11. *A matrix A is orthogonally diagonalizable ($D = Q^{-1}AQ \equiv Q^tAQ$) iff A is symmetric ($A = A^t$).*

3.4 Spectral decomposition

Let A be a real symmetric $n \times n$ matrix with evs $\lambda_1, \lambda_2, \dots, \lambda_n$ and corresponding orthonormal EVs $v_1, v_2 \dots, v_n$; then

$$\begin{aligned} A &= \begin{pmatrix} \vdots & \vdots & \vdots \\ v_1 & v_2 & v_3 \\ \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \lambda_1 & & \mathit{0} \\ & \ddots & \\ \mathit{0} & & \lambda_n \end{pmatrix} \begin{pmatrix} \dots & v_1 & \dots \\ \dots & v_2 & \dots \\ & \vdots & \\ \dots & v_n & \dots \end{pmatrix} \\ &= QDQ^t. \end{aligned}$$

This concludes the life and theory of a matrix in FM112.