Tutorial Worksheet-4 (WL5.1, WL5.2)
Orthogonal basis, properties of Orthonormal vectors, orthogonal projection and orthogonal complement, properties of orthogonal complement, advantage of orthogonal transformations, Gram-Schmidt process

Name and section: ____________________

Instructor's name: ____________________

1. Find the orthogonal projection $\vec{x}^{\parallel} = proj_v(\vec{x})$ of the vector $\vec{x} = (1, 2, 3)^T$, onto vector $\vec{v} = (-1, 0, 1)^T$.

Solution:

$$\begin{aligned}\vec{x}^{\parallel} &= \frac{\vec{x}.\vec{v}}{\vec{v}.\vec{v}}\vec{v} \\ \vec{x}^{\parallel} &= \frac{-1+0+3}{1+0+1}(-1,0,1)^t \\ \vec{x}^{\parallel} &= (-1,0,1)^t\end{aligned}$$

2. Find the orthogonal projection of $\begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ onto the subspace of $\mathbb{R}^4$ spanned by $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Solution: since $\left\{ \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ are the set of orthogonal vectors. let orthogonal projection of $9\vec{e_1}$ onto the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ is $\vec{x}$

so

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

since these vectors are orthogonal. hence

$$c_1 = \frac{\begin{bmatrix}9\\0\\0\\0\end{bmatrix} \cdot \begin{bmatrix}2\\2\\1\\0\end{bmatrix}}{\begin{bmatrix}2\\2\\1\\0\end{bmatrix} \cdot \begin{bmatrix}2\\2\\1\\0\end{bmatrix}} = \frac{18}{9} = 2$$

$$c_2 = \frac{\begin{bmatrix}9\\0\\0\\0\end{bmatrix} \cdot \begin{bmatrix}-2\\2\\0\\1\end{bmatrix}}{\begin{bmatrix}-2\\2\\0\\1\end{bmatrix} \cdot \begin{bmatrix}-2\\2\\0\\1\end{bmatrix}} = \frac{-18}{9} = -2$$

$$\vec{x} = 2\begin{bmatrix}2\\2\\1\\0\end{bmatrix} - 2\begin{bmatrix}-2\\2\\0\\1\end{bmatrix} = \begin{bmatrix}8\\0\\2\\-2\end{bmatrix}$$

hence $\begin{bmatrix}8\\0\\2\\-2\end{bmatrix}$ is the orthogonal projection of $\begin{bmatrix}9\\0\\0\\0\end{bmatrix}$ onto the subspace of $\mathbb{R}^4$ spanned by $\begin{bmatrix}2\\2\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-2\\2\\0\\1\end{bmatrix}$

3. Find an orthnormal basis for the space which is spanned by $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}2\\-2\end{bmatrix}\right\}$ in $\mathbb{R}^2$.

Solution: Let $v_1 = \begin{bmatrix}2\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}2\\-2\end{bmatrix}$
Let $\vec{\gamma_1} = \vec{v_1} = (2,1)^t$
Now, normalize $\vec{\gamma_1}$,
i.e

$$\vec{u_1} = \frac{\vec{\gamma_1}}{||\vec{\gamma_1}||} = \frac{(2,1)^t}{\sqrt{4+1}} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)^t$$

$$\vec{\gamma_2} = \vec{v_2} - (\vec{u_1}.\vec{v_2})\vec{u_1} = (2,-2)^t - \left(\frac{4}{\sqrt{5}} - \frac{2}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)^t = \left(\frac{6}{5}, \frac{-12}{5}\right)^t$$

Now, normalize $\vec{\gamma_2}$,
i.e

$$\vec{u_2} = \frac{\vec{\gamma_2}}{||\vec{\gamma_2}||} = \frac{\left(\frac{6}{5}, \frac{-12}{5}\right)^t}{\sqrt{\frac{36}{25} + \frac{144}{36}}} = \left(\frac{1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right)^t$$

hence the orthonormal basis is $\left\{ \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \right\}$

4. The set $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^3$. Use the Gram-Schmidt process to create an orthonormal basis of $\mathbb{R}^3$.

Solution: Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Let $\vec{\gamma_1} = \vec{v_1} = (1, 0, 0)^t$

Now, normalize $\vec{\gamma_1}$,

i.e

$$\vec{u_1} = \frac{\vec{\gamma_1}}{||\vec{\gamma_1}||} = \frac{(1,0,0)^t}{\sqrt{1+0+0}} = (1,0,0)^t$$

$$\vec{\gamma_2} = \vec{v_2} - (\vec{u_1}.\vec{v_2})\vec{u_1} = (1,1,1)^t - (1+0+0)(1,0,0)^t = (0,1,1)^t$$

Now, normalize $\vec{\gamma_2}$, i.e

$$\vec{u_2} = \frac{\vec{\gamma_2}}{||\vec{\gamma_2}||} = \frac{(0,1,1)^t}{\sqrt{0+1+1}} = \left(0, \frac{1}{2}, \frac{1}{2}\right)^t$$

$$\begin{aligned} \vec{\gamma_3} = \vec{v_3} - (\vec{u_1}.\vec{v_3})\vec{u_1} - (\vec{u_2}.\vec{v_3})\vec{u_2} &= (1,1,-1)^t - (1+0+0)(1,0,0)^t - \left(0 + \frac{1}{2} - \frac{1}{2}\right)\left(0, \frac{1}{2}, \frac{1}{2}\right) \\ &= (0,1,-1)^t \end{aligned}$$

Now, normalize $\vec{\gamma_3}$,

i.e

$$\vec{u_3} = \frac{\vec{\gamma_3}}{||\vec{\gamma_3}||} = \frac{(0,1,-1)^t}{\sqrt{0+1+1}} = \left(0, \frac{1}{2}, -\frac{1}{2}\right)^t$$

hence the orthonormal basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix} \right\}$