5|1|2022

Agenda items

- Diagonalization of matrices
- Similarity transformation
- Spectral decomposition of matrices

Last Lecture

→ evs & EVs (defⁿ, meaning, etc)

→ trace of a matrix & its relⁿ w/ evs; det ↔ evs.

Diagonalizable matrices

Certain forms of matrices are convenient to work with . . . . .

→ Upper/Lower triangular forms Why?

→ Diagonal forms Why?

think EVs,

think computing powers of matrices, . . .

Diagonalizable matrices . . .

Wouldn't it be nice if

$$\underset{\text{(any } n\times n \text{ matrix)}}{A} \longrightarrow \underset{\text{(diagonal form)}}{D}$$

the evs of A and D will be same.

$A \in M_{n\times n}(\mathbb{F})$ is "diagonalizable" over $\mathbb{F}$ if there exists an invertible matrix S over $\mathbb{F}$ s.t.

$$\left\{ \begin{array}{l} A = SDS^{-1} \text{ or equivalently} \\ D = S^{-1}AS \leftarrow \text{Similarity transformation} \end{array} \right.$$

Q) When is a matrix diagonizable?

Ans) $A \in \mathbb{M}_{n \times n}(\mathbb{F})$ is diagonalizable if and only if A has n linearly independent EVs in $\mathbb{F}^n$

* An n×n complex matrix that has n distinct evs is diagonalizable.

Example:

Q) Find a matrix that diagonalizes

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix}$$

Ans Solve $|A - \lambda I| = 0$ to obtain

Check if $S^{-1}AS$ is D?

$$\lambda_{1,2} = 3 \pm i$$

then use $A\vec{x}_j = \lambda_j \vec{x}_j$; $j = 1, 2$

to obtain $\vec{x}_1 = \begin{pmatrix} \frac{-1+i}{2} \\ 1 \end{pmatrix}$; $\vec{x}_2 = \begin{pmatrix} \frac{-1-i}{2} \\ 1 \end{pmatrix}$

$S = \begin{pmatrix} \frac{-1+i}{2} & -\frac{1-i}{2} \\ 1 & 1 \end{pmatrix}$ diagonalizes A.

$$S^{-1}AS = \begin{pmatrix} -i & \frac{1-i}{2} \\ +i & \frac{1+i}{2} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} \frac{-1+i}{2} & \frac{-1-i}{2} \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3+i & 0 \\ 0 & 3-i \end{pmatrix}$$

$$= D!$$

✓

* the col^{m} vectors of S form an eigenbasis for A & the diagonal entries of D are the associated eVs.

Ques) What are the evs and EVs of the n x n identity matrix I_n?

Is there an eigenbasis for I_n?

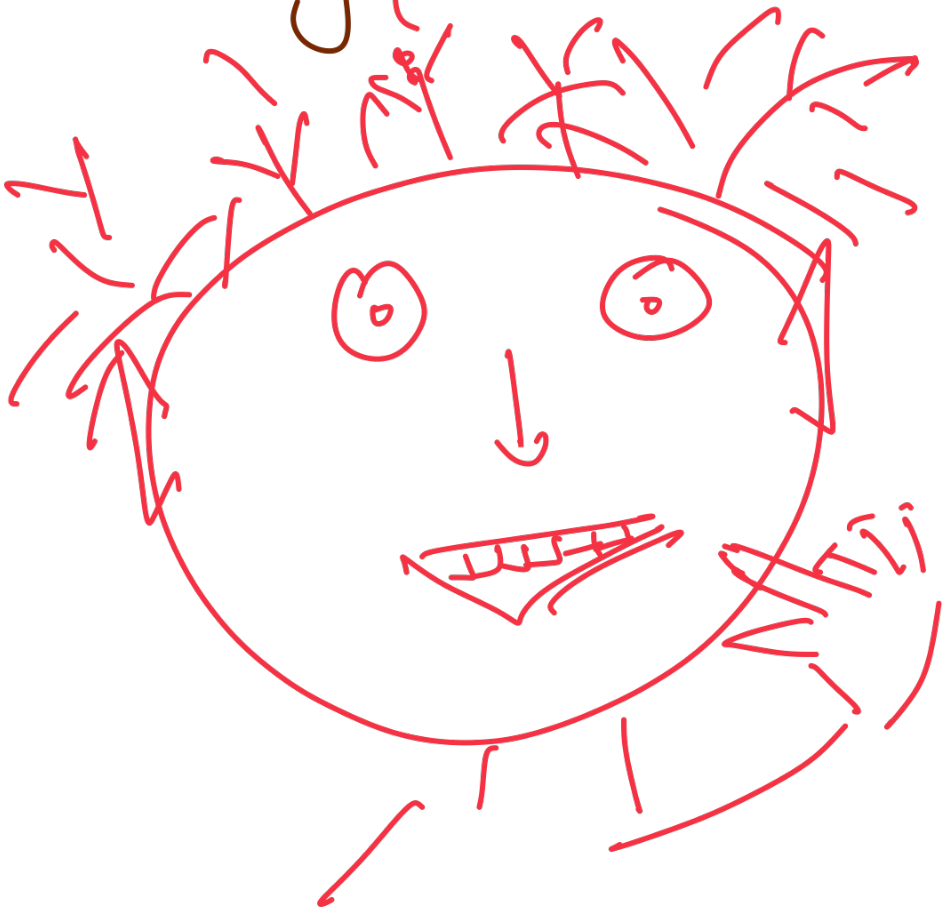
Which matrix diagonalizes I_n?

this is in some sense a silly & yet a conceptually trick question!!

Example

Q) Find the eigenspace of $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Is A diagonalizable ?



Ans: the eVs are along the diagonal.

$$\lambda_{1,2} = 1, 0$$

alg. mult. 1

alg. mult. 2

To find EVs :- $\vec{x}_1 = \ker(A - 1\,\mathbb{I}_3)$

$$= \ker\underbrace{\begin{pmatrix}0&1&1\\0&-1&1\\0&0&0\end{pmatrix}}_{\Lambda}$$

$$\overset{\mathrm{rref}(\Lambda)}{=} \ker\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix} = sp\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \vec{0}$$

$$\Rightarrow \begin{pmatrix}x_2\\x_3\\0\end{pmatrix} = \vec{0}$$

$\Rightarrow x_2, x_3 = 0$

and $x_1 = 1$ (actually any arbitrary const.)

Likewise $\vec{x}_2 = sp\begin{pmatrix}-1\\1\\0\end{pmatrix}$

b/c we are able to find only 2 indep. EVs!

∴ We cannot find S to diagonalize A!

By inspection $\vec{x}_1$ & $\vec{x}_2$ span only the x_1-x_2 plane

Geometric multiplicity of ev

$$\text{gemu}(\lambda) = \dim\left(\text{Ker}(A - \lambda \mathbb{I}_n)\right)$$
$$= \text{nullity}(A - \lambda \mathbb{I}_n)$$
$$= n - \text{rank}(A - \lambda \mathbb{I}_n)$$

In prev. example

$$\text{gemu}(1) = \dim\left(\text{Ker}(A - \mathbb{I}_3)\right) = \dim\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$= 1$$
$$\neq \text{almu}(1)$$
$$= 2.$$

* Spectral th^m

A matrix A is orthogonally diagonalizable ($D = Q^{-1} A Q \equiv Q^{\dagger} A Q$) iff A is symmetric ($A^{\dagger} = A$)

* Spectral decomposition

Let A be a real symmetric n×n matrix w/ EVs $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding orthonormal EVs $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$; then

$$A = \begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} \leftarrow \vec{v}_1 \rightarrow \\ \leftarrow \vec{v}_2 \rightarrow \\ \vdots \\ \leftarrow \vec{v}_n \rightarrow \end{pmatrix}$$

$$A = Q \qquad D \qquad Q^{\dagger}$$

this concludes the life
and theory of a Matrix
in FM112