DATE 19. 2. 2025

Lec-10

Let $A \in M_{m \times n}(\mathbb{R})$

So A has m rows, and n col^{m}s.

$\therefore$ Col(A) has vectors that are embedded in $\mathbb{R}^m$

What about AA^T?

$$AA^T \in M_{m \times m}(\mathbb{R})$$

Q) Now, what can you say about Col(A) and Col(AA^T)?

Recall, the col^m picture of $A\bar{u} = \bar{f}$

$$A\bar{u} = \begin{pmatrix} | & | & | & & | \\ \bar{v}_1 & \bar{v}_2 & \bar{v}_3 & \cdots & \bar{v}_n \\ | & | & | & & | \end{pmatrix}_{m \times n} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

$$= u_1 \begin{pmatrix} | \\ \bar{v}_1 \\ | \end{pmatrix} + u_2 \begin{pmatrix} | \\ \bar{v}_2 \\ | \end{pmatrix} + \cdots + u_n \begin{pmatrix} | \\ \bar{v}_n \\ | \end{pmatrix}$$

Now instead of $\bar{u}$ if we had $\mathbb{B} \in M_{n \times p}(\mathbb{R})$, how could we write $A\mathbb{B}$ as a col^m picture?

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We can consider the col^{m}s of $\mathbb{B}$ as a **tessallation** of $\bar{u}$ type vectors as follows

$$\mathbb{B} = \begin{pmatrix} u_1^{(1)} & u_1^{(2)} & & u_1^{(p)} \\ u_2^{(1)} & u_2^{(2)} & \cdots & u_2^{(p)} \\ \vdots & \vdots & & \vdots \\ u_n^{(1)} & u_n^{(2)} & & u_n^{(p)} \end{pmatrix}$$

$$= \begin{pmatrix} | & | & & | \\ \vec{u}^{(1)} & \vec{u}^{(2)} & \cdots & \vec{u}^{(p)} \\ | & | & & | \end{pmatrix}$$

Let's take a small example & see that the above is indeed what we expect!

$$\begin{pmatrix} 0 & 10 \\ 3 & 7 \\ 5 & 3 \end{pmatrix}_{3\times2} \begin{pmatrix} 0 & 3 & 5 \\ 10 & 7 & 3 \end{pmatrix}_{2\times3} = \begin{pmatrix} 100 & 70 & 30 \\ 70 & 58 & 36 \\ 30 & 36 & 34 \end{pmatrix}_{3\times3}$$

$\mathbb{A}$ $\quad$ $\mathbb{A}^T$

$$\| $$

$$\left[\underbrace{0\begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} + 10\begin{pmatrix} 10 \\ 7 \\ 3 \end{pmatrix}} \quad \underbrace{3\begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} + 7\begin{pmatrix} 10 \\ 7 \\ 3 \end{pmatrix}} \quad \underbrace{5\begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} + 3\begin{pmatrix} 10 \\ 7 \\ 3 \end{pmatrix}} \right]$$

$$= \begin{bmatrix} 100 & 70 & 30 \\ 70 & 58 & 36 \\ 30 & 36 & 34 \end{bmatrix}$$

lin comb^n of col^{m}s of $\mathbb{A}$ $\quad$ lin. comb^n of col^{m}s. of $\mathbb{A}$ $\quad$ lin comb^n of col^{m}s of $\mathbb{A}$

$\dim(\text{null}(A)) = \dim(\text{null}(AA^T))$

$\text{Col}(AA^T) \subseteq \text{Col}(A)$

$\dim(\text{Col}(AA^T)) \leq \dim(\text{Col}(A))$

$\text{rank}(AA^T) \leq \text{rank}(A)$

& Since $\text{Col}(\mathbb{A}) :=$ col of $\mathbb{A}$ & "all" their lin comb^n

$\text{rank}(AA^T) = \text{rank}(A)$

$$\boxed{\text{Col}(\mathbb{A}) \equiv \text{Col}(\mathbb{A}\mathbb{A}^T)}$$

Whence one can write

$$(\mathbb{A}\mathbb{B})_{m\times p} = \left[\underbrace{u_1^{(1)}\begin{pmatrix} | \\ \vec{v}_1 \\ | \end{pmatrix} + u_2^{(1)}\begin{pmatrix} | \\ \vec{v}_2 \\ | \end{pmatrix} + \cdots + u_n^{(1)}\begin{pmatrix} | \\ \vec{v}_n \\ | \end{pmatrix}}_{1^{st}\text{ col}^m\text{ of }\mathbb{A}\mathbb{B}} \quad \underbrace{u_1^{(2)}\begin{pmatrix} | \\ \vec{v}_1 \\ | \end{pmatrix} + u_2^{(2)}\begin{pmatrix} | \\ \vec{v}_2 \\ | \end{pmatrix} + \cdots + u_n^{(2)}\begin{pmatrix} | \\ \vec{v}_n \\ | \end{pmatrix}}_{2^{nd}\text{ col}^m\text{ of }\mathbb{A}\mathbb{B}} \quad \cdots \quad \underbrace{u_1^{(p)}\begin{pmatrix} | \\ \vec{v}_1 \\ | \end{pmatrix} + u_2^{(p)}\begin{pmatrix} | \\ \vec{v}_2 \\ | \end{pmatrix} + \cdots + u_n^{(p)}\begin{pmatrix} | \\ \vec{v}_n \\ | \end{pmatrix}}_{p^{th}\text{ col}^m\text{ of }\mathbb{A}\mathbb{B}} \right]$$

HW: Should the bases of $\text{Col}(A)$ and $\text{Col}(AA^T)$ be the same necessarily?? #