Systems of Ordinary Differential Equations (ODEs)

Example 1:

$$\frac{dx}{dt} = 2x - xy$$
$$\frac{dy}{dt} = -3y + 0.5xy$$

Coupled ODEs

Example 2:

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = -3y$$

Decoupled ODEs

Here, the equations can be solved separately, e.g.
$x(t) = c_1 e^{2t}$, $\mathrm{y}(t) = c_2 e^{-3t}$

Note that the solution of a system of two differential equations is a pair of functions $x(t)$ and $y(t)$ that simultaneously satisfy both equations

In this lecture, we study systems of **Autonomous First-Order ODEs in Two Variables**

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

Note that the RHS of these First-Order ODEs depend explicitly only on the dependent variables x and y and only implicitly on the independent variable t

In mathematics, an autonomous system or autonomous differential equation is a system of ordinary differential equations which does not explicitly depend on the independent variable. When the variable is time, these are also referred to as time-invariant systems.

Many laws in physics, where the independent variable is usually assumed to be time, are expressed as autonomous systems because it is assumed the laws of nature which hold now are identical to those for any point in the past or future, i.e. *the system will evolve in the same manner, regardless of when it starts* (*when we specify* t=0)

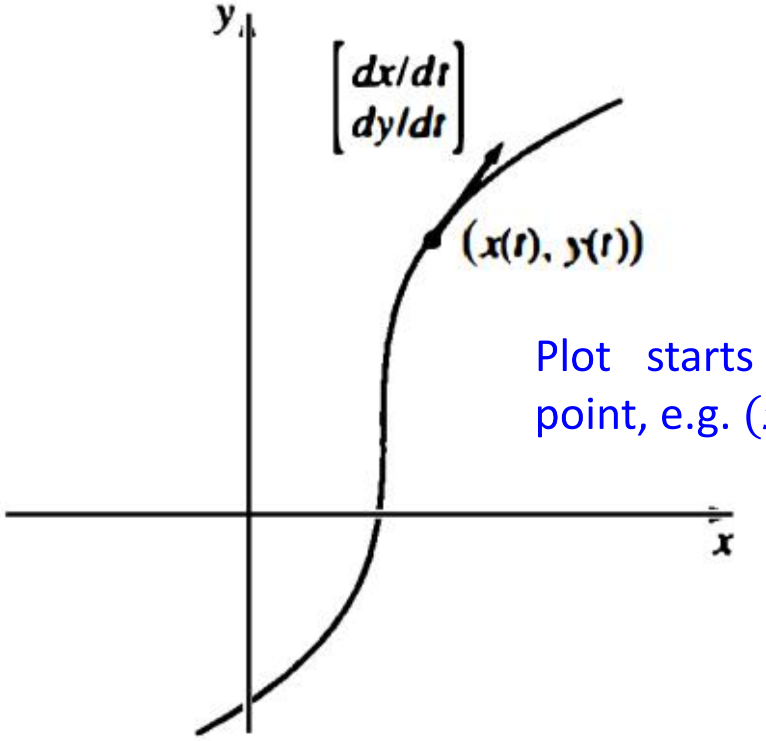
$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y)$$

The functions $x(r)$and $y(r)$ or $\begin{pmatrix} x(r) \\ y(r) \end{pmatrix}$ that satisfy these ODEs represent a parametric curve $(x(t), y(t))$, which can be plotted in the xy-plane.

Given a starting point $(x(0), y(0))$ as the initial condition, we can visualize the solution as a curve that has the correct tangent vector at each point where the tangent vector is –

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}, \quad \text{for } \frac{dx}{dt} \neq 0$$



Plot starts from some initial point, e.g. $(x(0), y(0)$ at $t = 0)$

The plot is a *trajectory* (curve) from the initial point with the correct tangent vector at all subsequent points

Some Definitions for a Phase Plane of a Ordinary Differential Equation System in Two Variables x and y -

1. Phase-Plane: The $x - y$ plane

2. Vector Field: The collection of *tangent vectors* defined by the ODE

3. Trajectory: The parametric curve defined by the solutions $\{x(t) \text{and } y(t)\}$ or, equivalently by the vector $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

4. States: The points (x, y) of the Phase-Plane

5. Phase Portrait: The collection of trajectories corresponding to different Initial Conditions (ICs)

Example (Drawing Phase-Plane Trajectories)

$$\frac{dx}{dt} = 2x$$

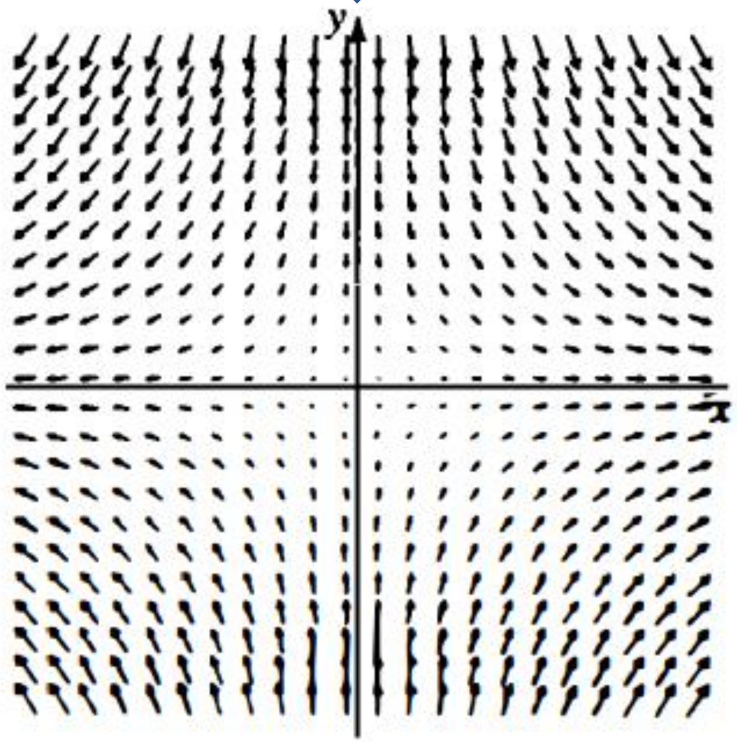
$$\frac{dy}{dt} = -3y$$

$$x(t) = c_1 e^{2t},\ y(t) = c_2 e^{-3t}$$

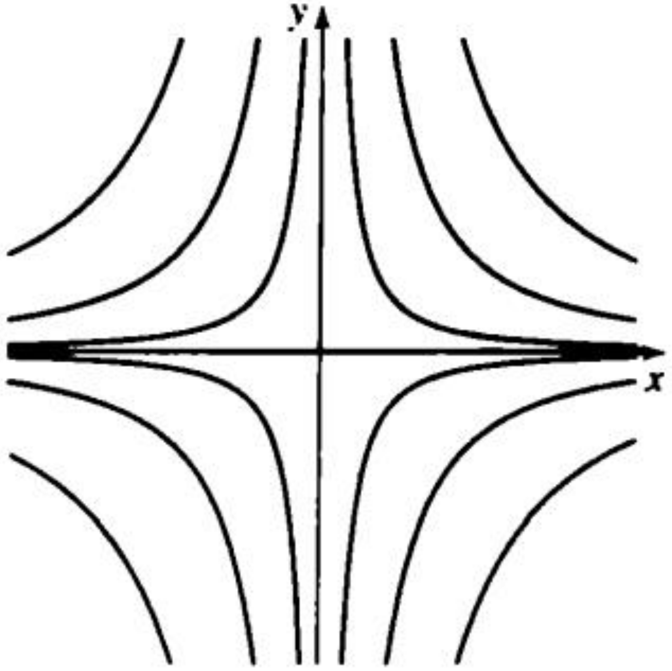
$$\Rightarrow \frac{dy}{dx} = \frac{-3y}{2x} = -\frac{3}{2}\frac{y}{x}$$

(x, y)	$\frac{dy}{dx}$
(1, 10)	-15
(1, 6)	-9
(1, 2)	-3
(2, 1)	- 0.75
(10, 1)	- 0.15

Some calculated test points



Vector Field



Solution Curves

Points (x, y) of the phase plane are called the **states** of the system and the collection of trajectories for various initial conditions is the **phase portrait** of the system

Equilibrium Points and Stability

An equilibrium point for a two-dimensional system is an (x, y) point, where $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ simultaneously.

Note that, if a state is at equilibrium then it does not change. However, an equilibrium point can be either stable or unstable

- A **stable** equilibrium *attracts* (or at least tries to keep close) solutions which are nearby.
- An **unstable** equilibrium *repels* nearby solutions in at least one direction

If the equilibrium is stable, minor perturbations to a system at equilibrium will tend to return it to the equilibrium state. On the contrary, if the equilibrium was unstable then this will force the system farther and farther away from the equilibrium point.

In the previous example, (x, y) =(0, 0) , i.e. the origin, is the only equilibrium point.

However, this is an unstable equilibrium as all the nearby trajectories are sent away (repelled) in the horizontal direction

Sketching Phase Plane Trajectories

Recall the "quiver" function of MATLAB, used to draw vectors.
You can use this to draw phase-plane trajectories as well.

Hand-graphing tool for *phase-plane analysis* uses ***Nullclines***, which is an adaptation of *isoclines*

An **isocline** of a differential equation $y' = f(t, y)$ is a curve in the t, y – plane along which the slope is constant. For example, the set of all points (t, y) where the slope y' has the value c, i.e. the graph of $f(t, y) = c$ would be the isocline for c.

Nullclines:

- A v *nullcline* is an isocline of vertical slopes, i.e. where $\frac{dx}{dt} = 0$
- An h *nullcline* is an isocline of horizontal slopes, i.e. where $\frac{dy}{dt} = 0$

Equilibria occurs at the points where a v nullcline intersects a h nullcline

Directions for Phase-Plane Trajectories

	$-$	0	+
$\frac{dx}{dt}$	$\longleftarrow$	$\vert$	$\longrightarrow$
$\frac{dy}{dt}$	$\downarrow$	$-$	$\uparrow$

Nullclines and Equilibria

- Where $dx/dt = 0$, the slopes are vertical
- Where $dy/dt = 0$, the slopes are horizontal
- Where both $dx/dt = 0$ and $dy/dt = 0$ $\Rightarrow$ Equilibrium Point

Left/Right Directions

- Where dx/dt is +ive, arrow points right
- Where dx/dt is -ive, arrow points left

Up/Down Directions

- Where dy/dt is +ive, arrow points up
- Where dy/dt is -ive, arrow points down

When existence and uniqueness conditions hold for the system, its *phase-plane trajectories cannot intersect*
(i.e. they may come close but will not cross each other)

Try once again to draw the phase plane trajectories of

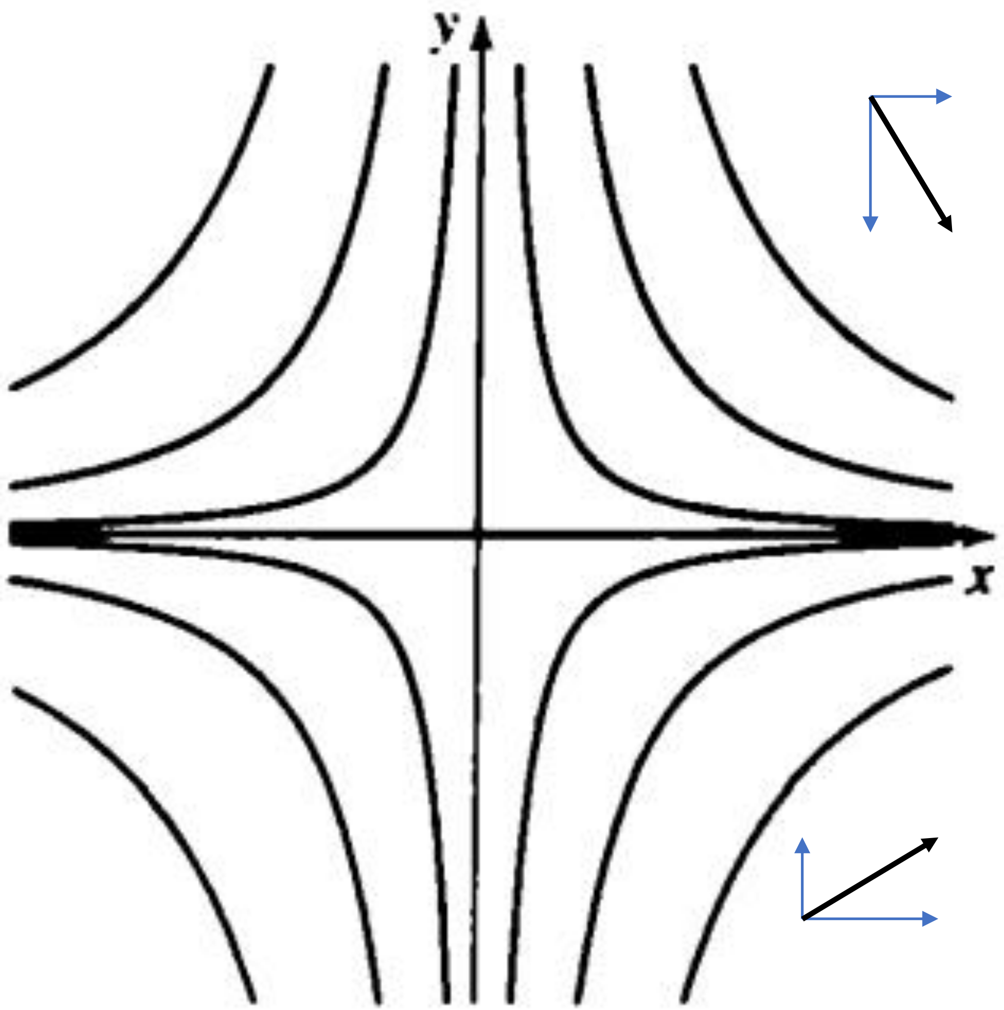
$$\frac{dx}{dt} = 2x$$

$$\frac{dy}{dt} = -3y$$

(i) Start with the equilibrium point (0, 0)

(ii) Take a few points in the phase plane. Sketch the v and h nullclines and then draw the resultant vector.

As you consider more and more points, the phase-plane trajectory will get drawn



Another Interesting Example

$$\frac{dx}{dt} = 1 - x - y \qquad \frac{dy}{dt} = 1 - x^2 - y^2$$

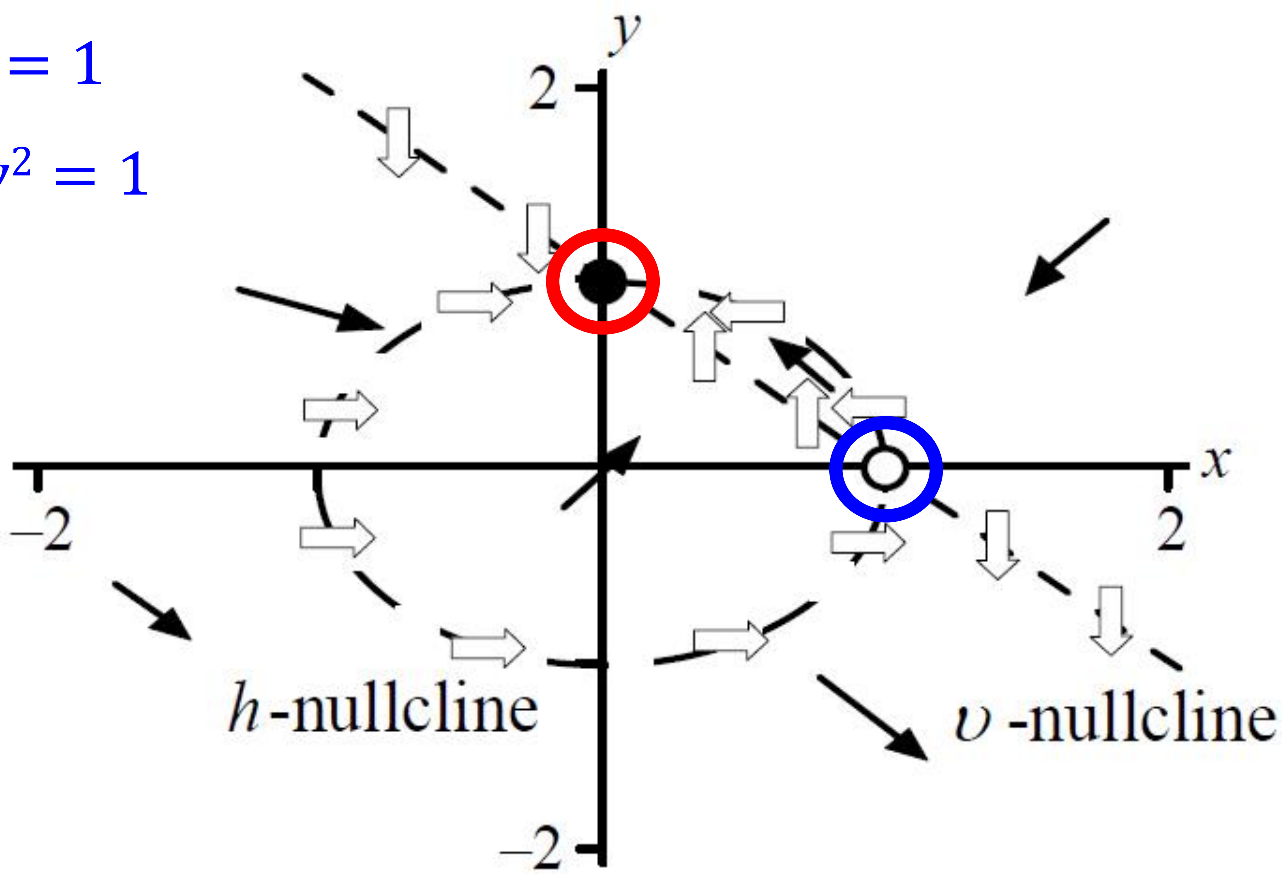
$\frac{dx}{dt} = 0$ gives the v-nullcline: $x + y = 1$

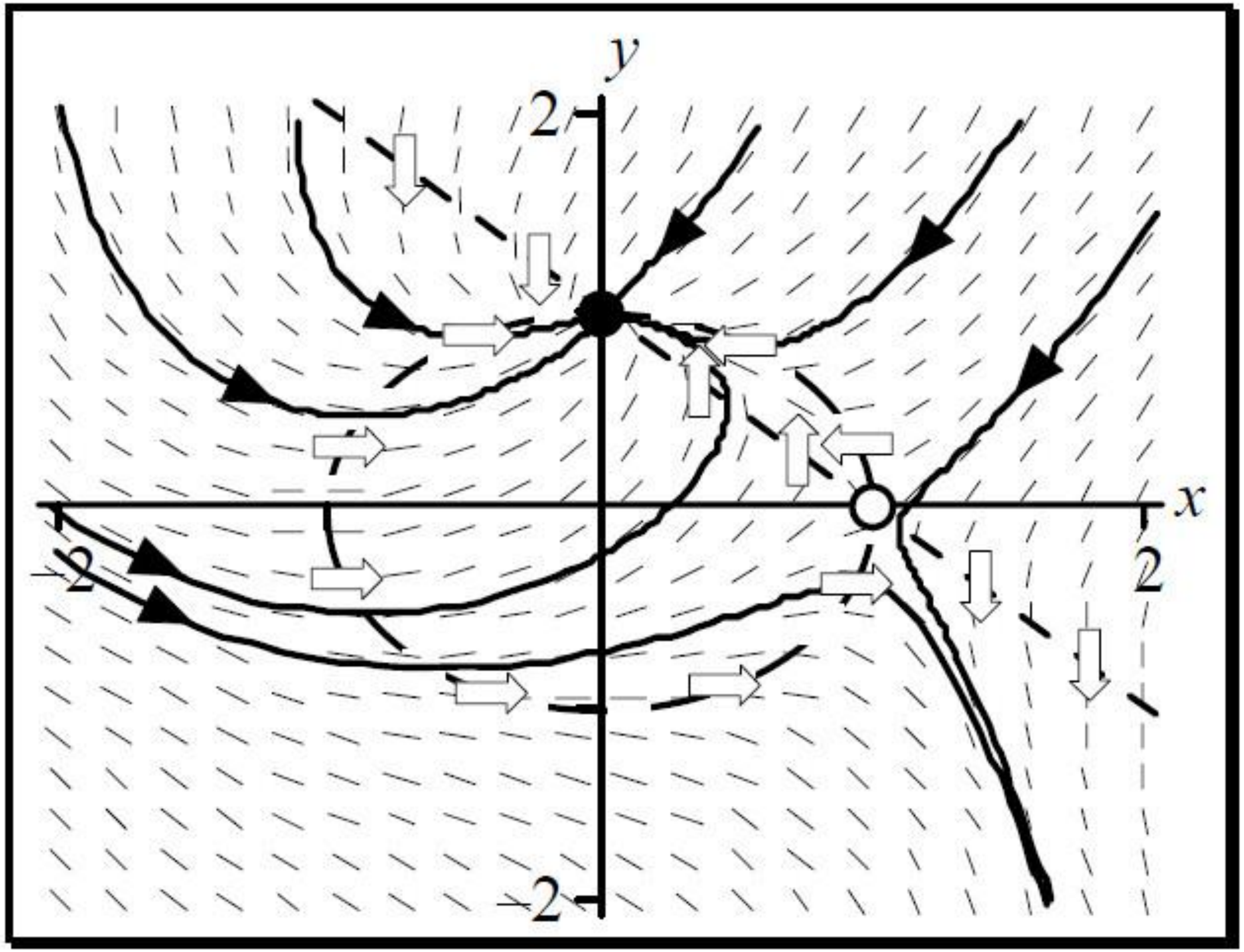
$\frac{dy}{dt} = 0$ gives the h-nullcline: $x^2 + y^2 = 1$

Two Equilibrium Points –

Stable Equilibrium Point at
$(x = 0, y = 1)$

Unstable Equilibrium Point at
$(x = 1, y = 0)$





Phase-Plan Trajectory

for -

$$\frac{dx}{dt} = 1 - x - y$$
$$\frac{dy}{dt} = 1 - x^2 - y^2$$

Notice the Trajectories converging on the stable equilibrium point (0, 1)