Solutions to systems of linear Differential eqs.

$$\vec{x}'(t) = A(t)\vec{x}(t) + \vec{f}(t) \quad \text{w/ ICs}$$

eg. $\begin{cases} x' = 3x - 2y \\ y' = x \\ z' = -x + y + 3z \end{cases}$

$$\vec{x}' = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{pmatrix} \vec{x}$$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

It may be verified easily that

$$\vec{x}_h = \begin{pmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}$$ is a soln by checking $\vec{x}_h' = A\vec{x}_h$

Infact it may be verified that $\begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}$ & $\begin{pmatrix} e^{t} \\ e^{t} \\ 0 \end{pmatrix}$ are also sol^{n}s to

Likewise the non-homogeneous ODE system

$$\begin{cases} x' = 3x - 2y + 2 - 2e^{t} \\ y' = x - e^{t} \\ z' = -x + y + 3z + e^{t} - 1 \end{cases}$$

$$\vec{x}' = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 3 \end{pmatrix}\vec{x} + \begin{pmatrix} 2 - 2e^{t} \\ -e^{t} \\ e^{t} - 1 \end{pmatrix}$$

This system has a particular soln.

$$\vec{x}_p = \begin{pmatrix} e^t \\ 1 \\ 0 \end{pmatrix}$$

Check by substituting in the system of ODE & verifying that indeed

$$\vec{x}_p' = A\vec{x}_p + \vec{f}(t) \qquad \vec{f}(t) = \begin{pmatrix} 2-2e^t \\ -e^t \\ e^t-1 \end{pmatrix}$$

So what is the full sol^n now?

By the "<u>superposition principle</u>" for homogeneous Linear ODE: (pg. 346, see 6.1).

$$\vec{X}(t) = \vec{X}_h + \vec{X}_p = c_1\begin{pmatrix}0\\0\\e^{3t}\end{pmatrix} + c_2\begin{pmatrix}2e^{2t}\\e^{2t}\\e^{2t}\end{pmatrix} + c_3\begin{pmatrix}e^{t}\\e^{t}\\0\end{pmatrix} + \underbrace{\begin{pmatrix}e^{t}\\1\\0\end{pmatrix}}_{\vec{X}_p}$$

as long as we can show that the 3 homogeneous sol^n's are linearly independent vectors.

Need to show $\vec{x}_1 = \begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix}$,

$\vec{x}_3 = \begin{pmatrix} e^{t} \\ e^{t} \\ 0 \end{pmatrix}$ are linearly independent on $(-\infty, \infty)$

Step 1:- Choose a pt. $t_0 = 0 \in (-\infty, \infty)$

Step 2:- Calculate $\vec{x}_1(t_0)$, $\vec{x}_2(t_0)$, $\vec{x}_3(t_0)$ and form the col^{m}. sp. matrix

$$C = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Step 3 :- Test for linear independence of col^{m}s. of C by computing .

$$\text{rref}(C) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Clearly the col^m vectors of C must be linearly independent

Could have also shown that $\det(C) \neq 0 \Rightarrow$ col^{m}s. of C are lin. indep.

Fundamental matrix

$\vec{x}_h$ can be expressed as

$$c_1\begin{pmatrix} 0 \\ 0 \\ e^{3t} \end{pmatrix} + c_2\begin{pmatrix} 2e^{2t} \\ e^{2t} \\ e^{2t} \end{pmatrix} + c_3\begin{pmatrix} e^{t} \\ e^{t} \\ 0 \end{pmatrix}$$

OR

$$\underbrace{\begin{pmatrix} 0 & 2e^{2t} & e^{t} \\ 0 & e^{2t} & e^{t} \\ e^{3t} & e^{2t} & 0 \end{pmatrix}}_{X(t)} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}}_{\vec{c}}$$

(i) $\det(X(t)) \neq 0$

(ii) The Fundamental Matrix is NOT unique, a diff. set of lin. independent solns. will produce a diff. $X(t)$ but $\vec{x}_h = X(t)\vec{c}$ would hold!

One can show $X'(t) = A\,X(t)$

How do we find $\vec{x}_h$ and $\vec{x}_p$ for a system of linear ODE?

First $\vec{x}_h$

Again we will have 3 main cases.

(i) Distinct real eVs

(ii) Repeated real eVs

(iii) Complex eVs

eVs of A!

$$\vec{x}\,' = A\vec{x}$$

Case (i) : $\vec{X}' = A\vec{X}$

has real evs $\lambda_1, \lambda_2, \ldots, \lambda_n$
$\lambda_i \neq \lambda_j \; \forall \; i \neq j$
& corresponding EVs $\vec{v}_1, \ldots, \vec{v}_n$

General homogeneous soln

$$\vec{X}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots\cdots + c_n e^{\lambda_n t}\vec{v}_n$$

Note :- in the case of repeated evs
$\lambda_i = \lambda_j$ for some $i \neq j$; we will
need either independent EVs or generalized EVs

eg. Solve the system of ODEs

$$\frac{dx}{dt} = -2x + y$$

$$\frac{dy}{dt} = x - 2y$$

; w/ $x(0) = 3$, $y(0) = 1$

Soln:- $\vec{x}\,' = \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}$ $\vec{x}$; $\vec{x}(0) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

eVs: $\lambda_1 = -1$, $\lambda_2 = -3$

EVs: $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

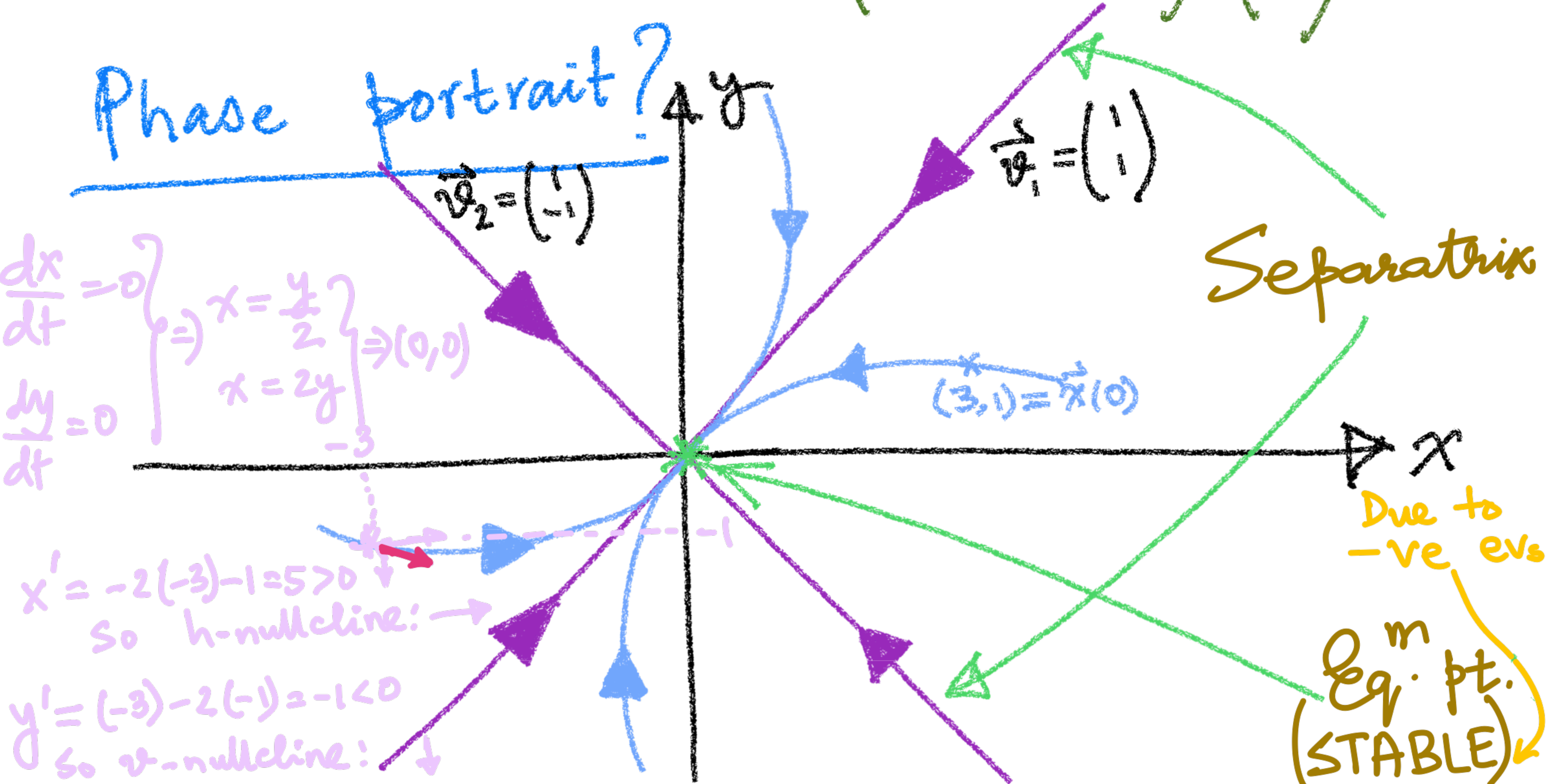
∴ the general soln: $\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

IC: $\vec{x}(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow c_1 = 2$, $c_2 = 1$

$\therefore$ the final general soln. is

$$\vec{x}(t) = X(t)\vec{c} = \begin{pmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Phase portrait?



Case (ii): $\vec{x}' = \underbrace{A}_{}\vec{x}$

$A \in M_{2\times 2}(\mathbb{R})$ for simplicity of expl^n.

repeated ev $\lambda_1 = \lambda_2 = \lambda$ w/ only one EV: $\vec{v}$

Construct an additional lin. indep. EV: $\vec{u}$

Step (i): find $\vec{v}$ corresponding to λ

Step (ii): find a new $\vec{u} \neq \vec{0}$ s.t.

$$(A - \lambda I)\vec{u} = \vec{v}$$

Step (iii): then $\vec{x}(t) = c_1 e^{\lambda t}\vec{v} + c_2 e^{\lambda t}(t\vec{v} + \vec{u})$

* $\vec{u}$ is known as the "generalized EV" of A corresponding to the ev λ.

eg. Solve the system:

$$\vec{x}' = A\vec{x} = \underbrace{\begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}}\vec{x}$$

ev: $\lambda = 4$ (repeated)

EV: $\vec{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

One solⁿ: $\vec{x}_1(t) = e^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Note: the 2^{nd} trial solⁿ from our 1^{st} lecture of module (3) i.e. $\vec{x}_2(t) = te^{4t}\vec{v}$ will NOT work !!

Try plugging in $\vec{x}_2(t)$ in $\vec{x}' = A\vec{x}$ & see if it satisfies the ODE!

(cf. pg. 363, sec 6.2)

So lets try the generalized EV $\vec{u}$

Whereby $\vec{x}_2(t) = e^{4t}(t\vec{v} + \vec{u})$ is a soln.

We must solve $\underbrace{(A-4I)}\vec{u} = \vec{v}$

$$\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} -2u_1 - u_2 = 1 \\ 4u_1 + 2u_2 = -2 \Rightarrow 2u_1 + u_2 = -1 \end{cases}$$

$\Rightarrow u_1 = k$ (say)

$u_2 = -2k - 1$

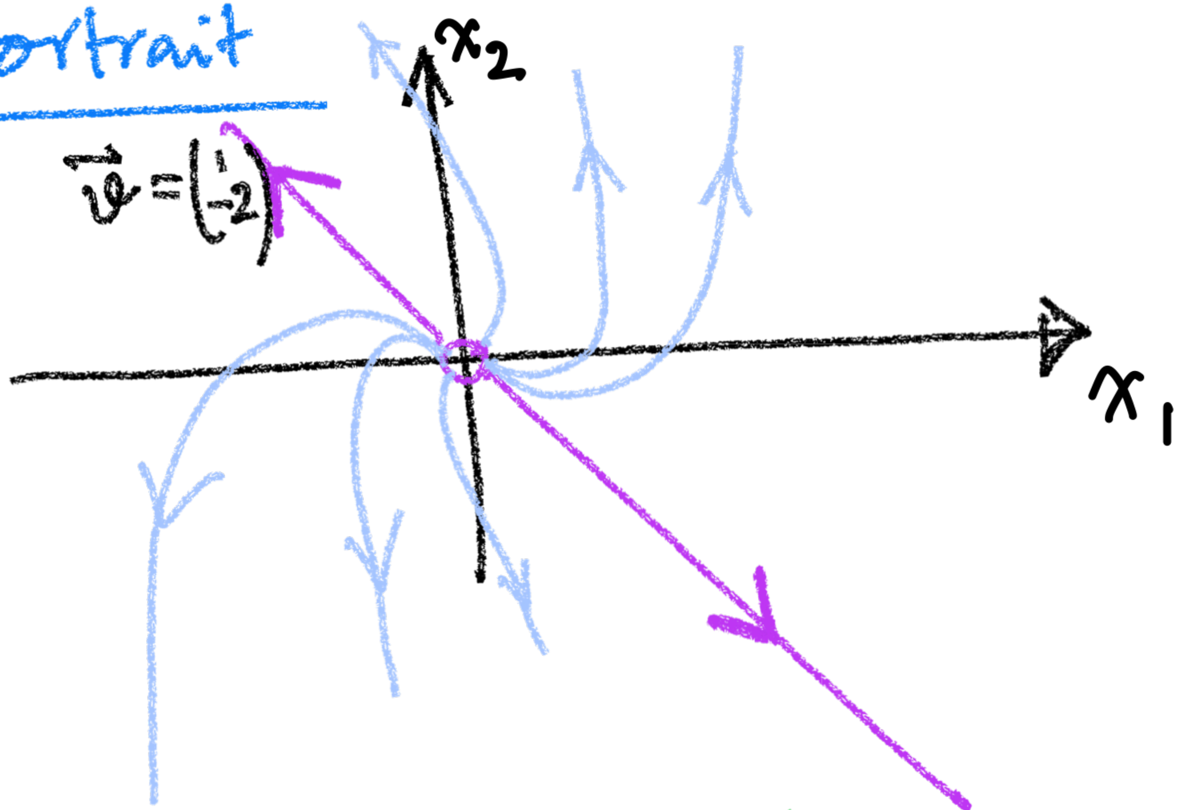
$\therefore \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = k\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

ignore b/c identical to $\vec{x}_1$

So $\vec{x}_2 = te^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + ke^{4t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + e^{4t}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = e^{4t}\begin{pmatrix} t \\ -2t-1 \end{pmatrix}$

$\therefore$ full soln :- $\vec{x}(t) = c_1 e^{4t}\begin{pmatrix}1\\-2\end{pmatrix} + c_2 e^{4t}\begin{pmatrix}t\\-2t-1\end{pmatrix}$

Phase portrait



(i) Unstable eq^m pt. at $(0,0)$ b/c of $\lambda > 0$

(ii) Only one separatrix $\vec{v}$

(iii) Why could we not sketch the other separatrix involving the generalized EV $\vec{u}$?

Next lecture:

(i) Complex eVs
(ii) Particular sol^{n}s. $\vec{X}_p$ for Systems of linear ODE
(iii) phase portrait & stability analysis