3/12/19

Lecture (14) :- Singularities in the Complex plane

Definitions :-

(14·1) **Singularity:** $z = z_0$ is a singular point of $f(z)$ if $f'(z_0)$ does not exist (i.e. $f(z)$ is not analytic at z_0) but $f'(z)$ exists in atleast one pt. which is in the neighborhood of z_0.

(14·2) **Isolated singular point !-** If $f(z)$ is analytic in the region $0 < |z - z_0| < R$ but not at $z = z_0$; then z_0 is called an isolated singular point.

In such a case, in the neighborhood of $z = z_0$, $f(z)$ may be represented by a Laurent Series

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n$$

* $f(z)$ would be analytic at z_0 if $C_0 \overset{\text{at}}{=} f(z_0)$, the power series converges at z_0.

(14·3) **Removable Singularity :-**

If $C_0 \neq f(z_0)$, then by a slight redefinition of $f(z_0)$, $f(z)$ is analytic there.

eg $f(z) = \dfrac{\sin z}{z} \Rightarrow$ at $z = 0$ $f(z)$ is not defined & hence not analytic

But since the power series expansion gives

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left\{ z - \frac{z^3}{3!} + \cdots \right\} = 1 - \frac{z^2}{3!} + \cdots$$

$C_0 = 1 \neq f(0)$. But if we redefine $f(0) = 1$; then $f(z)$ is analytic $\forall z$ (including $z = 0$)

In other words, if $f(z)$ is analytic in the region $0 < |z - z_0| < R$, and if $f(z)$ can be made analytic at $z = z_0$ by assigning an appropriate value for $f(z_0)$; then $z = z_0$ is a removable singularity.

(14.4) An isolated (POLE) singularity at z_0 of $f(z)$ is said to be a pole if $f(z)$ has the form $f(z) = \frac{\phi(z)}{(z - z_0)^N}$; $N \in \mathbb{Z}^+$, $N \geq 1$ & $\phi(z)$ is analytic in the neighborhood of $z = z_0$ & $\phi(z_0) \neq 0$

N^{th} order pole for $N \geq 2$

simple pole when $N = 1$

(14.5) Strength of the pole (C_{-N})

$$f(z) = \frac{\phi(z)}{(z - z_0)^N} \underset{\text{expansion}}{\overset{\text{Laurent}}{=}} \sum_{n=-\infty}^{\infty} \frac{C_n (z - z_0)^n}{(z - z_0)^N}$$

but b/c $\phi(z)$ is analytic in the n'hood of $z = z_0$

Laurent series ≡ Taylor series

$$\sum_{n=0}^{\infty} C_n (z - z_0)^{n-N}$$

$$\overset{m=n-N}{=} \sum_{m=-N}^{\infty} C_m (z - z_0)^m$$

$$= C_{-N}(z - z_0)^{-N} + \sum_{m=-N+1}^{\infty} C_m (z - z_0)^m$$

$$\Rightarrow \underbrace{f(z)(z - z_0)^N}_{\phi(z)} = C_{-N} + C_{-N+1}(z - z_0) + \cdots$$

$$\Rightarrow \boxed{\phi(z_0) = C_{-N}}$$ is the strength of the pole of $f(z)$.

(14.6) Essential singular point

An "isolated" singular pt. that is neither removable nor a pole is called an essential singular pt. Such a pt. has full Laurent series expansion.

eg $e^{1/z}$ @ $z = 0$.

Note :- Entire f^ns are either constant f^ns or at $z = \infty$ they have isolated singular pts or essential singularities.

Examples

eg (14.1) Describe the singularities of the f^n:

$$f(z) = \frac{z^2 - 2z + 1}{z(z+1)^3} = \frac{(z-1)^2}{z(z+1)^3}$$

Ans

the f^n $f(z)$ has a simple pole at $z = 0$ & a 3^{rd} order (triple) pole at $z = -1$

Strength of the pole at $z = 0$ is 1 b/c the expansion of $f(z)$ near $z = 0$ has the form

$$f(z) = \frac{1}{z}(1 - 2z + \cdots)(1 - 3z + \cdots)$$

$$= \frac{1}{z} - 5 + \cdots \quad \text{so } c_{-1} = 1$$

Alternatively for the pole $z = 0$; $\varphi(z) = \frac{(z-1)^2}{(z+1)^3}$ which is analytic in the neighborhood of $z = 0$ $\therefore c_{-1} = \varphi(0) = 1$

Likewise for the pole $z = -1$

$$\varphi(z) = \frac{(z-1)^2}{z}$$ which is analytic in the neighborhood of $z = -1$

$$\therefore C_{-1} = \varphi(z=-1) = \varphi(-1) = \frac{(-1-1)^2}{-1} = -4.$$

this can also be checked from the Laurent series of $f(z)$ near $z = -1$

$$f(z) = \frac{-4}{(z+1)^3}.$$

eg (14.2) Describe the singularities of the f^n

$$f(z) = \frac{z+1}{z \sin z}$$

Soln:- using Taylor series for $\sin z$

$$f(z) = \frac{z+1}{z\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)} = \frac{(z+1)}{z^2\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right)}$$

$$= \frac{z+1}{z^2}\left\{1 + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right) + \left(\frac{z^2}{3!} - \frac{z^4}{5!} + \cdots\right)^2 + \cdots\right\}$$

$$= \left(\frac{1}{z^2} + \frac{1}{z}\right)\left(1 + \frac{z^2}{3!} + \cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + \cdots$$

$\Rightarrow f(z)$ has a double pole at $z = 0$ w/ strength $C_{-1} = 1$.

eg (14.3) Discuss the pole singularities of the f^n

$$f(z) = \frac{\log(z+1)}{(z-1)}$$

Soln :- $f(z)$ is multivalued w/ a branch pt. at $z=-1$,
therefore we make $f(z)$ single valued by introducing a branch cut from $z=-1$ to $z=\infty$ along the -ve real axis w/ $z = re^{i\theta}$; $-\pi \le \theta < \pi$;
this branch fixes $\log(1) = 0$.
W/ this choice of branch, $f(z)$ has a simple pole at $z=1$ w/ strength $\log(2)$.

A branch point is an example of a non-isolated singular pt. b/c a circuit around the b.p. results in a discontinuity via the removal of the branch cut.
$\therefore$ $z = -1$ is a b.p. & not a pole b/c $\log z$ has a jump discontinuity as we encircle $z=-1$. It is not analytic in a neighborhood of $z=-1$ $\Rightarrow$ $z=-1$ is not an isolated s.p.

eg (14.4) Discuss the pole singularities of the f^n.

$$f(z) = \frac{z^{1/2} - 1}{z - 1}$$

Soln:- $z = 1 + t$

$$\Rightarrow f(z) = \frac{\pm\sqrt{1+t} - 1}{t}$$

Where $\pm$ denotes the 2 branches of the sq. root f^n w/ $\sqrt{x} \geq 0$ for $x \geq 0$. ($z = 0$ is a sq. root b.p.)

$$\sqrt{1+t} \overset{\text{taylor}}{=} 1 + \frac{t}{2} - \frac{t^2}{8} + \cdots$$

thus for "+" branch

$$f(z) = \frac{t/2 - t^2/8 + \cdots}{t} = \frac{1}{2} - t/8 + \cdots$$

& for "−" branch

$$f(z) = \frac{-2 - \frac{t}{2} + \frac{t^2}{8} - \cdots}{t} = \frac{-2}{t} - \frac{1}{2} + \frac{t}{8} \cdots$$

$\therefore$ On the "+" principal branch, $f(z)$ is analytic in the neighborhood of $t = 0$ $\Rightarrow$ $t = 0$ is a removable singularity.

On the "−" principal branch, $t = 0$ is @ simple pole w/ strength $= -2$.

Def^n (14.7) Cluster point is a singular pt. (S.pt) in which an ∞ - sequence of isolated s.p.s cluster abt a pt. ($z = z_0$) in such a way that there are an ∞ - no. of isolated s.p. in any arbitrarily small circle about $z = z_0$. There is no Laurent series represent^n valid in the neighborhood of a cluster pt.

eg. $f(z) = \tan(1/z)$

as $z \to 0$ along the real axis, $\tan(\frac{1}{z})$ has poles at locations

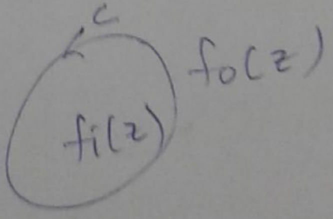
$$z_n = \frac{1}{\pi/2 + n\pi}, \; n \in \mathbb{Z}$$

which cluster b/c any small neighborhood of the origin contains an ∞ no. of them.

Def^n (14.8) Bdy jump discontinuity

eg. $$f(z) = \frac{1}{2\pi i} \oint_C \frac{1}{\zeta - z}\, d\zeta = \begin{cases} f_i(z) = 1 \; ; \; w/i \; C \\ f_o(z) = 0 \; ; \; w/o \; C \end{cases}$$

$f_i(z) \neq f_o(z)$ on C.



Def^n (14.9) Meromorphic f^{n}s. There are f^{n}s that are everywhere analytic in the finite $\mathbb{C}$ except at isolated pts. where they have poles. Such f^{n}s have only poles in the finite z - plane. they may have essential singularities at ∞ (like entire f^{n}s.).